N.M. GÜNTER

POTENTIAL THEORY

AND ITS APPLICATIONS TO BASIC PROBLEMS OF MATHEMATICAL PHYSICS

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FOREWORD TO THE RUSSIAN EDITION

The present book is the translation of N. M. GÜNTER's monograph "La théorie du potentiel et ses applications aux problèmes fondamentaux de la physique mathématique" published in Paris in 1934. The work arose from a special seminar on potential theory held by the author during the early twenties at Leningrad University.

Potential theory and problems of mathematical physics thereto related have been focal points of the mathematician's interest since the beginning of the 19th century. At first the properties of the different potentials were not subjected to rigorous investigation, and there were thus various unfounded results in applying potential theory to boundary value problems of mathematical physics. On the other hand, up to the end of the 19th century there were no definite, deep-seated results on the properties of the solutions of these problems near the boundary.

The well-known work of LYAPUNOV "Sur certaines questions qui se rattachent au problème de DIRICHLET" (1898) was of fundamental importance in overcoming these deficiencies. To a certain extent, GÜNTER'S book is closely connected with the results of this work. His objective was a precise and more detailed study of the properties of various potentials and the boundary-value problems of mathematical physics involving them.

In the translation the book was somewhat altered. In certain places the presentation was made more precise, several long proofs were simplified, and new material was incorporated to bring the content of the book up to the present level of the subject.

The supplements and changes consist mainly in the following:

In the second chapter an investigation of the potential of a double layer in the case in which the density satisfies a HÖLDER condition was added. In addition, the definition of the generalized LAPLACE operator according to I. I. PRIVALOV and a corresponding theorem were incorporated. Several important additions were also made at the conclusion of this chapter. The properties of the potential of mass occupying a certain region of space, the potential of the simple layer, and the potential of the double layer with smooth densities and boundaries were studied; under the same smoothness assumptions, this was followed by an investigation of the properties of the

direct values of the double-layer potential and the normal derivative of the simple-layer potential. The proof of the properties mentioned is given in the Appendix. Moreover, certain new results on properties possessed by the potential of mass in space, the potential of the simple layer, and the potential of the double layer with summable density under the hypothesis that the boundary of the region is a LYAPUNOV surface are presented.

In the third chapter the proof of the applicability of GREEN's formula to the potential of a simple layer with continuous density was altered. The proof of the uniqueness of the solution of the NEUMANN problem taken from a paper of M. V. KELDYSH and M. A. LAVRENTEV was added as was an investigation of the properties of the solution of the NEUMANN problem for smooth boundary values and surfaces.

In the fourth chapter only one addition was made. This is concerned with the study of the properties of the solutions of the DIRICHLET problem under the smoothness assumptions mentioned above.

The majority of changes occur in the fifth chapter which contains the theory of GREEN's functions and integral equations related to boundary-value problems for the wave equation and the heat equation. The proof at the beginning of the chapter that GREEN's function can be represented by the potential of a simple layer was altered. A number of estimates for GREEN's function and the function of F. NEUMANN were added. The section "GREEN's Function and POISSON's Equation" was inserted. In this section those functions are studied which admit an integral representation with a GREEN or NEUMANN function as kernel. In connection with this, the study of the equation $\triangle u = Lu + K$ was altered somewhat. The presentation of the main theorem on expansion in terms of eigenfunctions and that of the theorem of V. A. STEKLOV were also changed. New results were presented on the solution of the boundary-value problem for the wave equation. At the end of the fifth chapter a supplement on the properties of eigenfunctions for regions with smooth boundaries was incorporated.

GÜNTER presented the complicated proofs of a number of theorems in four appendices at the end of the book. In the present edition the proof in Appendix I has been simplified by making use of previous results. The material of Appendix III has been basically changed. Appendix IV of the old text, which treated the problem of closedness in the class of bounded and square-integrable functions, has been left out, since its content was taken up in the main text. In its place the direct values of the double-layer potentials on smooth surfaces are investigated in Appendix IV of the present edition.

The changes and supplements were made by Kh. L. SMOLITSKII.

V. I. SMIRNOV

CONTENTS

CHAPTER I

LEMMATA

§1. On the Boundary of Regions

We shall always assume that the regions considered are bounded by a finite number of closed surfaces which satisfy the three LYAPUNOV *conditions*:

1. At each point of the surface there exists a well-defined tangent plane and hence a well-defined normal.

2. If ϑ is the angle between the normals at the points M_1 and M_2 and r is the distance between these points, then

$$\vartheta < Er^{\lambda} \quad (0 < \lambda \leq 1), \tag{1}$$

where E and λ are constants.

3. For all points M of the surface there exists a single fixed number d with the property that the portion of the surface inside a sphere of radius d about M intersects lines parallel to the normal at M in at most one point.

If d is given, then any number smaller than d has the same property, so that we may still adjust d in a manner appropriate to our purpose. The sphere mentioned we shall call the LYAPUNOV *sphere.*

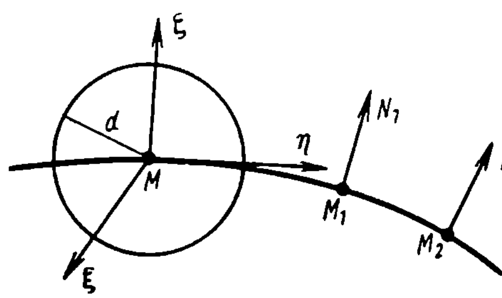

Fig. 1

At a point M of the surface we take the normal at that point as the ζ axis and place the ξ and η axes in the tangent plane at M (Fig. 1). Because of condition 3, the portion of the surface lying inside the LYAPUNOV sphere about M may be represented in the form

$$\zeta = \Phi(\xi, \eta), \tag{2}$$

where $\Phi(\xi, \eta)$ is a single-valued function in a certain region (Λ) of the $\xi\eta$ plane.

Condition 1 implies that in (Λ) the function $\Phi(\xi, \eta)$ possesses first derivatives with respect to ξ and η which because of 2 are continuous.

We shall henceforth denote the surface by (S). Let φ be the angle that the normal to (S) at a certain point a distance r from M makes with the ζ axis; let d be chosen such that $Ed^\lambda < 1$. Then for points of (S) inside the LYAPUNOV sphere:

$$\varphi < E r^\lambda < E d^\lambda < 1$$

and hence

$$\cos\varphi \geq 1 - \frac{\varphi^2}{2} > \frac{1}{2}.$$

On the other hand

$$\cos\varphi = \frac{1}{\sqrt{1 + (\Phi'_\xi)^2 + (\Phi'_\eta)^2}},$$

and therefore

$$\sqrt{(\Phi'_\xi)^2 + (\Phi'_\eta)^2} = \tan\varphi = \frac{\sin\varphi}{\cos\varphi} \leq \frac{\varphi}{1 - \frac{\varphi^2}{2}} < \frac{E r^\lambda}{1 - \frac{1}{2} E^2 r^{2\lambda}},$$

i.e.

$$\sqrt{(\Phi'_\xi)^2 + (\Phi'_\eta)^2} < \frac{E r^\lambda}{1 - \frac{1}{2} E^2 r^{2\lambda}} < \frac{E d^\lambda}{1 - \frac{1}{2} E^2 d^{2\lambda}}. \tag{3}$$

Lemma. *If d satisfies the inequality*

$$E d^\lambda < \frac{1}{2}, \tag{4}$$

then the intersection of the portion of (S) *contained in the* LYAPUNOV *sphere with an arbitrary plane containing the* ζ *axis is a continuous curve.*

Proof. First note that from formulas (3) and (4) it follows that

$$\sqrt{(\Phi'_\xi)^2 + (\Phi'_\eta)^2} < \frac{E r^\lambda}{1 - \frac{1}{2} \cdot \frac{1}{4}} = \frac{8}{7} E r^\lambda < \frac{8}{7} E d^\lambda < \frac{4}{7}. \tag{5}$$

Let the plane pass through the ξ axis and consider the curve of intersection for $\xi \geq 0$. The curve passes through the point M and leaves the

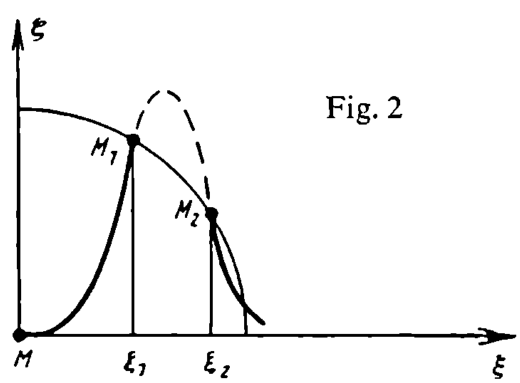

Fig. 2

sphere for the first time at a point M_1 with abscissa ξ_1 (Fig. 2). We shall show that no point of the curve with abscissa greater than ξ_1 can lie inside the sphere.

Let us assume that such points do exist, and let M_2 be the one such point with smallest abscissa which we denote by ξ_2. Now the angle which the tangent to the curve at M_2 makes with the ξ axis is not less than the angle between the tangent to the meridian curve of the LYAPUNOV sphere at M_2 and the ξ axis.

Hence we have

$$|\Phi'_\xi(\xi_2, 0)| \geq \frac{\xi_2}{\sqrt{d^2 - \xi_2^2}} > \frac{\xi_2}{d}.$$

Therefore

$$\xi_2 < d\,|\Phi'_\xi(\xi_2, 0)|$$

and also

$$\xi_1 < d\,|\Phi'_\xi(\xi_2, 0)|. \tag{6}$$

On the other hand

$$\zeta_1 = \Phi(\xi_1, 0) - \Phi(0,0) = \xi_1 \Phi'_\xi(\Theta\xi_1, 0) \qquad (0 < \Theta < 1)$$

and hence

$$d = \sqrt{\xi_1^2 + \zeta_1^2} = \xi_1\sqrt{1 + (\Phi'_\xi(\Theta\xi_1, 0))^2}. \tag{7}$$

From (7), (6), and (5) it follows that

$$d < d|\Phi'_\xi(\xi_2, 0)|\sqrt{1 + (\Phi'_\xi(\Theta\xi_1, 0))^2} < d\cdot\frac{4}{7}\sqrt{1 + \frac{16}{49}} = d\cdot\frac{4\sqrt{65}}{49} < d,$$

which is impossible. Hence the assumption that such a point M_2 exists leads to a contradiction, and the lemma is herewith proved.

From (7) and (5) we further obtain:

$$\xi_1 = \frac{d}{\sqrt{1 + (\Phi'_\xi(\Theta\xi_1, 0))^2}} > \frac{d}{\sqrt{1 + \frac{16}{49}}} = \frac{7}{\sqrt{65}}d > \frac{7}{9}d,$$

i.e., the region (Λ) is starlike with respect to the point M and contains a disk of radius $7/9\ d$ with M as origin.[1]

Theorem. *If the surface* (S) *satisfies the three* LYAPUNOV *conditions and condition* (4), *then there exists a number* ω *with the property that any line forming an angle with the normal at* M *which is less than* ω *can neither intersect the portion of* (S) *lying inside the* LYAPUNOV *sphere about* M *in more than one point nor be tangent to it.*

The lemma actually shows that the surface (S) divides the LYAPUNOV sphere into two parts; we shall call one part the upper part and the other the lower part. At the point of intersection a line which cuts the surface (S)

[1] A set S in R^n is said to be *starlike* with respect to one of its points p if for any point p' in S and any number λ in the interval (0, 1) the point $p'' = (1 - \lambda)\,p + \lambda p'$ is in S (Trans.).

either passes from one part of the sphere to the other or else is tangent to (S). A line cannot pass from one part of the sphere to the other without intersecting the surface (S).

Proof of the Theorem. Let the normal to (S) be directed into the upper part of the sphere, and let $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ be the direction cosines of a line which is tangent to (S) or at least cuts (S) twice inside the LYAPUNOV sphere. We first show that there exists a point (ξ_0, η_0) in (Λ) such that

$$\cos\gamma - \Phi'_\xi(\xi_0, \eta_0)\cos\alpha - \Phi'_\eta(\xi_0, \eta_0)\cos\beta = 0. \tag{8}$$

Indeed, if the line is tangent to the surface (S) then equation (8) holds at the point of tangency. If the line cuts the surface (S) twice, then the left side of equation (8) has different signs at the two successive points of intersection, for at one of the points of intersection the line passes from the lower to the upper part of the sphere and at the other from the upper to the lower part; hence, at the first point the line makes an acute angle with the normal to (S), and at the second point it makes an obtuse angle. Equation (8) is therefore satisfied at some point (ξ_0, η_0) of the region (Λ), since the left side is continuous in (Λ) and assumes there values of opposite sign. Using (8) it follows from the BUNYAKOVSKII-SCHWARZ inequality[2] that

$$\begin{aligned}\cos^2\gamma &= (\Phi'_\xi\cos\alpha + \Phi'_\eta\cos\beta)^2 \le (\cos^2\alpha + \cos^2\beta)((\Phi'_\xi)^2 + (\Phi'_\eta)^2)\\ &= (1-\cos^2\gamma)((\Phi'_\xi)^2 + (\Phi'_\eta)^2)\end{aligned}$$

and hence

$$\cos^2\gamma \le \frac{(\Phi'_\xi)^2 + (\Phi'_\eta)}{1 + (\Phi'_\xi)^2 + (\Phi'_\eta)^2},$$

so from (3)

$$|\cos\gamma| < \frac{Er^\lambda}{\sqrt{1 + \frac{1}{4}E^4 r^{4\lambda}}} < \frac{Ed^\lambda}{\sqrt{1 + \frac{1}{4}E^4 d^{4\lambda}}}. \tag{9}$$

Let

$$\omega = \arccos\frac{Ed^\lambda}{\sqrt{1 + \frac{1}{4}E^4 d^{4\lambda}}}. \tag{10}$$

Then a line making an angle with the ζ axis less than ω can neither be tangent to the surface (S) nor intersect it in more than one point inside the LYAPUNOV sphere.

Let us pass a line forming an angle ω with the ζ axis through the point M; the cone generated by rotating this line about the ζ axis we shall henceforth refer to as the (2ω)-*cone* (Fig. 3).

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[2] This inequality also goes by the name of the CAUCHY-SCHWARZ-BUNYAKOVSKII inequality with one or more of the prefixed names omitted (Trans.).

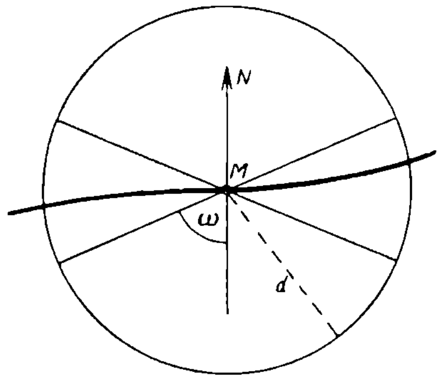

Fig. 3

As d tends to zero, ω tends to $\frac{\pi}{2}$. We shall subsequently always assume that ω is greater than $\frac{\pi}{3}$. This is the case when inequality (4) is satisfied.

Now suppose that a Cartesian coordinate system (x,y,z) has been arbitrarily chosen. Since

$$\cos^2(Nx) + \cos^2(Ny) + \cos^2(Nz) = 1,$$

one of the angles Nx, Ny, or Nz is less than $\frac{\pi}{3}$. It follows therefore that one of the axes of the coordinate system (x,y,z) forms an angle with the normal N which is less than ω, so that lines parallel to this axis are neither tangent to (S) nor intersect (S) in more than one point inside the LYAPUNOV sphere. It follows that inside each LYAPUNOV sphere the equation of the surface may be written in at least one of the following three forms:

$$z = F(x, y) \quad \text{or} \quad x = F(y, z) \quad \text{or} \quad y = F(z, x). \tag{11}$$

It should be noted that if inequality (4) holds a Cartesian coordinate system with the point M as origin can be chosen such that all three coordinate axes lie within the (2ω)-cone.

We further note that the angle which the normal at an arbitrary point of the surface inside the LYAPUNOV sphere makes with the normal at the center of the sphere does not exceed the value Ed^λ. This angle can be made arbitrarily small by choosing a sufficiently small d; if for example $Ed^\lambda < 0.017$, then this angle is less than $1°$, while the angle ω is greater than $89°$.

We end this section with certain estimates which we shall subsequently —mainly from Chapter II on—often use.

Let M be a point of the surface inside a LYAPUNOV sphere about M_0. The line M_0M then meets the surface inside the sphere in two points and therefore forms an angle α with the normal at M_0 which is greater than ω.

If we denote the length of the segment M_0M by r and by ϱ the projection of this segment onto the tangent plane at M_0, then

$$\varrho = r \sin \alpha > r \sin \omega,$$

and hence

$$\varrho \leq r < \frac{\varrho}{\sin \omega} < 2\,\varrho\,. \tag{12}$$

It then follows from inequality (5) that

$$\sqrt{(\Phi_\xi')^2 + (\Phi_\eta')^2} < \left(\frac{8\,E}{7\sin^\lambda \omega}\right) \varrho^\lambda = a\,\varrho^\lambda. \tag{13}$$

Moreover

$$\zeta = \Phi(\xi, \eta) = \Phi(\xi, \eta) - \Phi(0,0) = \xi \Phi_\xi'(\Theta\xi, \Theta\eta) + \eta\Phi_\eta'(\Theta\xi, \Theta\eta) \; (0 < \Theta < 1).$$

Using (13) and the BUNYAKOVSKII-SCHWARZ inequality we find

$$|\zeta| \leq \sqrt{\xi^2 + \eta^2}\,\sqrt{(\Phi_\xi'(\Theta\,\xi\,,\,\Theta\,\eta))^2 + (\Phi_\eta'(\Theta\,\xi\,,\,\Theta\,\eta))^2} < \varrho\,\sqrt{a^2 \varrho^{2\lambda}\Theta^{2\lambda}} < a\,\varrho^{1+\lambda},$$ 2

i.e.

$$|\zeta| < a\,\varrho^{1+\lambda}. \tag{14}$$

Let N be the normal at an arbitrary point of the surface (S); we shall assume that it is directed toward the same side of the surface as the ζ axis. Because of (5), the following inequality then holds for a point inside the LYAPUNOV sphere:

$$\cos(N\zeta) = \frac{1}{\sqrt{1 + (\Phi_\xi')^2 + (\Phi_\eta')^2}} > \frac{1}{\sqrt{1 + \frac{16}{49}}} > \frac{1}{2},$$

i.e.

$$\cos(N\zeta) > \tfrac{1}{2}. \tag{15}$$

Let $d\sigma^{(0)}$ be the projection onto the tangent plane $\xi\eta$ of the element $d\sigma$ of the surface (S). Introducing polar coordinates ϱ, φ in the $\xi\eta$ plane, we find using (15)

$$d\,\sigma = \frac{d\,\sigma^{(0)}}{\cos(N\,\zeta)} < 2\,d\,\sigma^{(0)} = 2\,\varrho\,d\varrho\,d\varphi\,. \tag{16}$$

It follows finally from (9) that if r is the distance between two points of (S) and N is the normal at the first of these points, then the line joining these two points intersects the portion of the surface (S) inside a LYAPUNOV sphere about the first point at least twice; hence

$$|\cos(rN)| = |\cos\gamma| < E r^\lambda,$$

i.e.,

$$|\cos(rN)| < E r^\lambda. \tag{17}$$

§2. On Functions Defined in the Interior of a Region

Surfaces which satisfy the three conditions of §1 we shall call LYAPUNOV *surfaces.*

We shall always assume that the boundary of the region (D) under consideration consists of a finite number of closed LYAPUNOV surfaces. We partition the continuous functions defined in the interior of a region (D) into three classes: a function $f(M) = f(x,y,z)$ is called

a) *continuous* in (D) if for every positive number ε there exists for each point M_0 a number h_0 such that

$$|f(M) - f(M_0)| < \varepsilon, \qquad \text{whenever} \qquad |MM_0| < h_0;$$

b) *uniformly continuous* in (D) if the number h_0 does not depend on the location of the point M_0; it is then equal to a number h which depends only on ε;

c) *H-continuous* in (D)—or we say that the function satisfies a HÖLDER *condition* in (D)—if for every pair of points M_0 and M a distance r apart,

$$|f(M) - f(M_0)| < Ar^\lambda, \qquad (0 < \lambda \leq 1)$$

where A and λ are fixed constants which are independent of M_0 and M. An H-continuous function is obviously uniformly continuous.

Theorem. *If the function $f(M)$ is uniformly continuous in the interior of the region (D), then $f(M)$ has a definite boundary value as M tends to a point M_0 of the boundary of (D).*

Proof. There exists an h_0 such that for any pair of points M_1 and M_2 of (D) with $|M_1M_2| < h_0$

$$|f(M_1) - f(M_2)| < 1.$$

We denote by $\varphi(h)$ the upper bound of the numbers $|f(M_1) - f(M_2)|$, where M_1 and M_2 are any two points of the region (D) such that $|M_1M_2| < h$ with $0 < h < h_0$. Obviously $0 \leq \varphi(h) \leq 1$, and $\varphi(h)$ is a nondecreasing function of h. Since the function $f(M)$ is uniformly continuous, $\varphi(h) \to 0$ for $h \to 0$. From the definition of $\varphi(h)$,

$$|f(M_1) - f(M_2)| \leq \varphi(h), \quad \text{whenever} \quad |M_1M_2| \leq h.$$

Now let M_0 be a boundary point of the region (D). The distance between any two points M_1 and M_2 of the region (D) lying inside a sphere of radius $\frac{h}{2}$ about M_0 is then less than h, and hence

$$|f(M_1) - f(M_2)| \leq \varphi(h).$$

From this it follows that when $M \to M_0$ the function $f(M)$ has a boundary value which we denote by $\bar{f}(M_0)$. If in the last inequality we let $M_2 \to M_0$ we obtain:

$$\left|f(M_1) - \bar{f}(M_0)\right| \leq \varphi(h), \text{ whenever } |M_1 M_0| < \frac{h}{2}\cdot$$

On (S) the boundary value $\bar{f}(M_0)$ is a *uniformly continuous* function of the points of (S). For if M_0 and M_1 are two points of the surface (S) a distance r apart, then the distance between the point M_1 and an arbitrary point M_2 in the interior of the region (D) and inside a sphere of radius $2r$ with M_0 as origin is at most $3r$; hence

$$\left|\bar{f}(M_1) - f(M_2)\right| \leq \varphi(6r),$$

and we thus obtain as $M_2 \to M_0$ that

$$\left|\bar{f}(M_1) - \bar{f}(M_0)\right| \leq \varphi(6r),$$

wherewith the assertion is proved.

If $f(M)$ is *H-continuous* in (D), then $\varphi(r) \leq Ar^\lambda$, and we thus obtain

$$\left|\bar{f}(M_1) - \bar{f}(M_0)\right| \leq A(6r)^\lambda = 6^\lambda A r^\lambda = Br^\lambda.$$

Hence the boundary values at points of (S) of a function H-continuous in (D) constitute an *H-continuous* function on the surface (S). We note that the function $f(M)$ is bounded if it is uniformly continuous in a finite region (D).

Theorem. *If the first derivatives of the function $f(M)$ are bounded in (D), then f is H-continuous in (D).*

Proof. If the points M_1 and M_2 can be joined by a line which does not intersect (S), then the assertion is a direct consequence of the mean value theorem in which

$$\lambda = 1, \qquad A = B\sqrt{3}$$

and B is an upper bound for the values of the first derivatives of f. If the distance between M_1 and M_2 is less than $\frac{d}{2}$ and if the distance of each of these points from the boundary is greater than $\frac{d}{2}$, then they can be joined by a line which does not intersect (S); hence the theorem is true for these two points.

We now assume that the distance between M_1 and M_2 is less than $\frac{d}{2}$, and the distance from the point M_1 to the boundary is not greater than $\frac{d}{2}$. M_1

then lies on the normal to (S) at the point M_0 of the boundary nearest M_1, and the distance between M_0 and M_2 is less than d; both points M_1 and M_2 are hence situated inside the LYAPUNOV sphere with origin at M_0.

Let us consider the (2ω)-cone with apex at M_0. If M_2 lies inside the cone the theorem is proved, since then M_1 and M_2 can be joined by a segment lying entirely in (D). It remains to consider the case in which M_2 does not lie inside the (2ω)-cone. We then pass a line through the point M_2 parallel to the axis of the cone; let M' be the point of intersection of this line with the surface of the cone (Fig. 4).

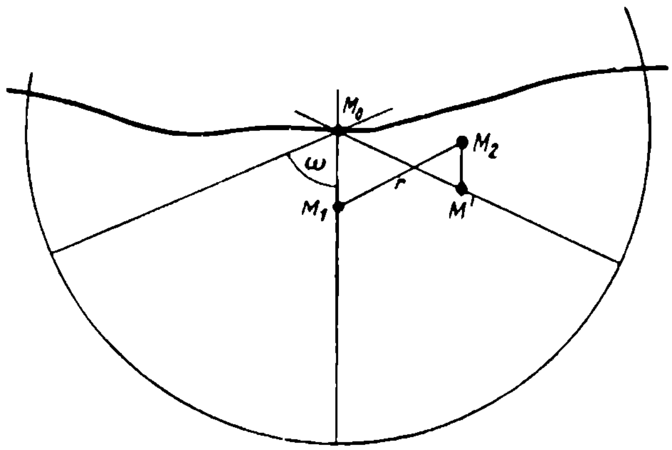

Fig. 4

The point M' lies inside the LYAPUNOV sphere about M_0. Its distance from the axis of the cone is in fact not greater than $|M_1M_2| < \frac{d}{2}$, so that the following inequality holds:

$$|M'M_0| \leq \frac{|M_1M_2|}{\sin\omega} < \frac{d}{2} : \frac{\sqrt{3}}{2} = \frac{d}{\sqrt{3}} < d.$$

Since the angle δ between M_1M' and M_2M' is greater than ω,

$$\frac{|M_1M'|}{|M_1M_2|} = \frac{\sin\beta}{\sin\delta} < \frac{1}{\sin\omega}, \quad |M_1M'| < \frac{r}{\sin\omega};$$

here r is the distance between M_1 and M_2, and β is the angle between M_1M_2 and M_2M'. Similarly,

$$|M_2M'| < \frac{r}{\sin\omega}.$$

From this it follows that

$$|f(M_1) - f(M_2)| \leq |f(M_1) - f(M')| + |f(M') - f(M_2)| < 2\sqrt{3}\,B \cdot \frac{r}{\sin\omega}.$$

§3. The HUGONIOT-HADAMARD Theorem

Differentiation of Functions Defined on a Surface

We assume that the function $f(x,y,z)$ has uniformly continuous first derivatives in the interior of (D). Under these conditions $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$ have definite boundary values as M tends to a point M_0 of the surface (S). We denote these boundary values by $\left(\frac{\partial f}{\partial x}\right)$, $\left(\frac{\partial f}{\partial y}\right)$, and $\left(\frac{\partial f}{\partial z}\right)$. We join two points M_1 and M_2 in the interior of (D) by the curve

$$x = \varphi(t), \quad y = \psi(t), \quad z = \chi(t), \tag{18}$$

of which it is assumed that the functions $\varphi(t)$, $\psi(t)$, and $\chi(t)$ possess first derivatives. On the curve (18) f is a function of t which we denote by $\bar{f}(t)$; we have

$$\frac{d\bar{f}}{dt} = \frac{\partial \bar{f}}{\partial x}\varphi'(t) + \frac{\partial \bar{f}}{\partial y}\psi'(t) + \frac{\partial \bar{f}}{\partial z}\chi'(t). \tag{19}$$

We now suppose that the curve (18) lies on the boundary. On this curve (f), $\left(\frac{\partial f}{\partial x}\right)$, $\left(\frac{\partial f}{\partial y}\right)$, and $\left(\frac{\partial f}{\partial z}\right)$ are functions of t.

Lemma. *If the curve (18) lies on the boundary of (D), then*

$$\frac{d(f)}{dt} = \left(\frac{\partial f}{\partial x}\right)\varphi'(t) + \left(\frac{\partial f}{\partial y}\right)\psi'(t) + \left(\frac{\partial f}{\partial z}\right)\chi'(t). \tag{20}$$

Proof. In proving the lemma we may assume that the arc M_1M_2 of the curve is so small that it is contained in a LYAPUNOV sphere with origin at the point M_1.

Let M' be a point of the curve (18) corresponding to parameter value t. Through M' we pass a line parallel to the normal at M_1 and lay off along this line a segment of length η in the interior of (D) (Fig. 5). The geometrical description of the endpoints of this segment is the curve

$$\left.\begin{aligned} x &= \varphi(t) + \eta\cos(Nx),\\ y &= \psi(t) + \eta\cos(Ny),\\ z &= \chi(t) + \eta\cos(Nz). \end{aligned}\right\} \tag{21}$$

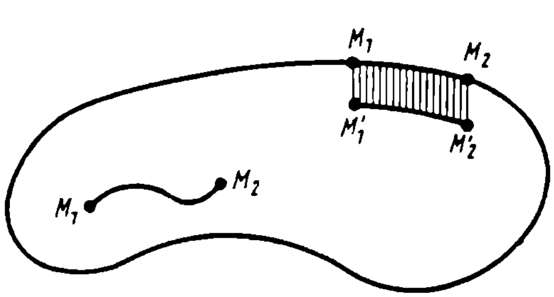

Fig. 5

Since formula (19) is valid for the curve (21), we obtain:

$$\frac{d\bar{f}}{dt} = f'_x[\varphi(t) + \eta\cos(Nx),\ \psi(t) + \eta\cos(Ny),\ \chi(t) + \eta\cos(Nz)]\,\varphi'(t) + \cdots.$$

From this it follows by integration that

$$\bar{f}_1 - \bar{f} = \int_t^{t_1} \{f'_x[\varphi(t') + \eta\cos(Nx),\ \psi(t') + \eta\cos(Ny),$$
$$\chi(t') + \eta\cos(Nz)]\,\varphi'(t') + \cdots\}\,dt',$$

where we denote by $\bar{f}_1$ the value of $\bar{f}$ at the point $t = t_1$. Since the function under the integral sign tends to a limit uniformly as η goes to zero, we find by passing to the limit under the integral sign that

$$(f_1) - (f) = \int_t^{t_1} \left\{\left(\frac{\partial f}{\partial x}\right)\varphi'(t') + \left(\frac{\partial f}{\partial y}\right)\psi'(t') + \left(\frac{\partial f}{\partial z}\right)\chi'(t')\right\} dt',$$

where (f_1) and (f) are the boundary values of f_1 and f. Dividing the last equation by $t_1 - t$ and then letting t_1 tend to t, we obtain equation (20).

We now suppose that we are given two functions f and F defined in the regions inside a LYAPUNOV sphere bounded by a portion of the surface and the parts of the sphere; the functions need not be identical as long as they are defined in the same region, and they furthermore have the property that

$$f = F \quad \text{on} \quad (S). \tag{22}$$

We further assume that in their domain of definition the functions f and F possess uniformly continuous first derivatives,

$$\frac{\partial f}{\partial x},\quad \frac{\partial f}{\partial y},\quad \frac{\partial f}{\partial z};\quad \frac{\partial F}{\partial x},\quad \frac{\partial F}{\partial y},\quad \frac{\partial F}{\partial z},$$

which on (S) assume the values

$$\left(\frac{\partial f}{\partial x}\right),\quad \left(\frac{\partial f}{\partial y}\right),\quad \left(\frac{\partial f}{\partial z}\right);\quad \left(\frac{\partial F}{\partial x}\right),\quad \left(\frac{\partial F}{\partial y}\right),\quad \left(\frac{\partial F}{\partial z}\right).$$

Theorem (HUGONIOT-HADAMARD). *At all points of* (S) *the following relation holds*:

$$\frac{\left(\frac{\partial f}{\partial x}\right) - \left(\frac{\partial F}{\partial x}\right)}{\cos(Nx)} = \frac{\left(\frac{\partial f}{\partial y}\right) - \left(\frac{\partial F}{\partial y}\right)}{\cos(Ny)} = \frac{\left(\frac{\partial f}{\partial z}\right) - \left(\frac{\partial F}{\partial z}\right)}{\cos(Nz)}. \tag{23}$$

Proof. We assume that (18) is an arbitrary curve on (S). Since $(f) - (F) = 0$ at all points of the curve, the lemma implies that

$$\frac{d[(f) - (F)]}{dt} = \left[\left(\frac{\partial f}{\partial x}\right) - \left(\frac{\partial F}{\partial x}\right)\right]\varphi'(t) + \left[\left(\frac{\partial f}{\partial y}\right) - \left(\frac{\partial F}{\partial y}\right)\right]\psi'(t)$$
$$+ \left[\left(\frac{\partial f}{\partial z}\right) - \left(\frac{\partial F}{\partial z}\right)\right]\chi'(t) = 0;$$

since the curve (18) lies on (S),

$$\varphi'(t)\cos(Nx)+\psi'(t)\cos(Ny)+\chi'(t)\cos(Nz)=0.$$

Eliminating $\chi'(t)$ from the last two equations, we obtain

$$\left\{\left[\left(\frac{\partial f}{\partial x}\right)-\left(\frac{\partial F}{\partial x}\right)\right]\cos(Nz)-\left[\left(\frac{\partial f}{\partial z}\right)-\left(\frac{\partial F}{\partial z}\right)\right]\cos(Nx)\right\}\varphi'(t)$$
$$+\left\{\left[\left(\frac{\partial f}{\partial y}\right)-\left(\frac{\partial F}{\partial y}\right)\right]\cos(Nz)-\left[\left(\frac{\partial f}{\partial z}\right)-\left(\frac{\partial F}{\partial z}\right)\right]\cos(Ny)\right\}\psi'(t)=0.$$

Since the values of $\varphi'(t)$ and $\psi'(t)$ may be chosen arbitrarily, it follows that

$$\left[\left(\frac{\partial f}{\partial x}\right)-\left(\frac{\partial F}{\partial x}\right)\right]\cos(Nz)-\left[\left(\frac{\partial f}{\partial z}\right)-\left(\frac{\partial F}{\partial z}\right)\right]\cos(Nx)=0,$$
$$\left[\left(\frac{\partial f}{\partial y}\right)-\left(\frac{\partial F}{\partial y}\right)\right]\cos(Nz)-\left[\left(\frac{\partial f}{\partial z}\right)-\left(\frac{\partial F}{\partial z}\right)\right]\cos(Ny)=0,$$

whence formula (23) follows immediately.

We shall agree to call the expression

$$\frac{df}{dn}=\left(\frac{\partial f}{\partial x}\right)\cos(Nx)+\left(\frac{\partial f}{\partial y}\right)\cos(Ny)+\left(\frac{\partial f}{\partial z}\right)\cos(Nz) \tag{24}$$

the normal derivative of the function f. If we choose a point M_2 on the normal to S at the point M_1 and determine the limit of the expression

$$\frac{f(M_2)-f(M_1)}{|M_1M_2|} \qquad (M_2\to M_1),$$

we obtain (24). It is easily seen that relations (23) are equivalent to the equations

$$\frac{\left(\frac{\partial f}{\partial x}\right)-\left(\frac{\partial F}{\partial x}\right)}{\cos(Nx)}=\frac{df}{dn}-\frac{dF}{dn},\ \ldots$$

We hence obtain the following expressions equivalent to (23):

$$\left(\frac{\partial f}{\partial x}\right)-\left(\frac{\partial F}{\partial x}\right)=\left(\frac{df}{dn}-\frac{dF}{dn}\right)\cos(Nx),$$
$$\left(\frac{\partial f}{\partial y}\right)-\left(\frac{\partial F}{\partial y}\right)=\left(\frac{df}{dn}-\frac{dF}{dn}\right)\cos(Ny),$$
$$\left(\frac{\partial f}{\partial z}\right)-\left(\frac{\partial F}{\partial z}\right)=\left(\frac{df}{dn}-\frac{dF}{dn}\right)\cos(Nz)$$

or

$$\left.\begin{aligned}\left(\frac{\partial f}{\partial x}\right)-\frac{df}{dn}\cos(Nx)&=\left(\frac{\partial F}{\partial x}\right)-\frac{dF}{dn}\cos(Nx),\\ \left(\frac{\partial f}{\partial y}\right)-\frac{df}{dn}\cos(Ny)&=\left(\frac{\partial F}{\partial y}\right)-\frac{dF}{dn}\cos(Ny),\\ \left(\frac{\partial f}{\partial z}\right)-\frac{df}{dn}\cos(Nz)&=\left(\frac{\partial F}{\partial z}\right)-\frac{dF}{dn}\cos(Nz).\end{aligned}\right\} \tag{25}$$

The left sides of equations (25) are independent of the choice of the function F, and the right sides are independent of the choice of the function f. Since the two functions are related only through condition (22), it follows that the left and right sides of equations (25) depend respectively only on the values of f and F on (S).

If a function μ is given on (S), if the function f satisfies the above conditions, and if in addition

$$(f) = \mu,$$

then we define three functions $D_x\mu$, $D_y\mu$, and $D_z\mu$ by the following equations:

$$\left.\begin{aligned} D_x\mu &= \left(\frac{\partial f}{\partial x}\right) - \frac{df}{dn}\cos(Nx), \\ D_y\mu &= \left(\frac{\partial f}{\partial y}\right) - \frac{df}{dn}\cos(Ny), \\ D_z\mu &= \left(\frac{\partial f}{\partial z}\right) - \frac{df}{dn}\cos(Nz). \end{aligned}\right\} \tag{26}$$

Because of the conditions imposed on the function f, $D_x\mu$, $D_y\mu$, and $D_z\mu$ are continuous functions on (S).

One should note that the quantities (26) are not independent: at each point the equation

$$\cos(Nx)D_x\mu + \cos(Ny)D_y\mu + \cos(Nz)D_z\mu = 0$$

holds, which shows that the vector with coordinates $D_x\mu$, $D_y\mu$, and $D_z\mu$ lies in the tangent plane of the surface.

Let us study this vector more closely. We consider the tangent plane at a certain point M_0 of the surface (S). In a suitable neighborhood of the point M every point P of the tangent plane is the projection of exactly one point M of the portion of (S) lying inside a LYAPUNOV sphere about M_0. A function $\mu(P)$ is defined in the tangent plane in terms of the function μ given on the surface (S) by putting $\mu(P) = \mu(M)$. If we define the function f such that it is equal to $\mu(P)$ at points of the tangent plane and takes on identical values along a perpendicular to the plane, then at the point M_0 the derivative $\frac{df}{dn}$ is equal to zero, and hence

$$D_x\mu = \left(\frac{\partial f}{\partial x}\right), \quad D_y\mu = \left(\frac{\partial f}{\partial y}\right), \quad D_z\mu = \left(\frac{\partial f}{\partial z}\right).$$

The vector under consideration is therefore equal to the gradient of the function f at the point M_0. With our choice of the function f this gradient coincides with the gradient of the function $\mu(P)$ which we consider to be defined in the tangent plane.

If for a given function μ on (S) the quantities $D_x\mu$, $D_y\mu$, and $D_z\mu$ exist, we shall say that the function μ is differentiable on (S). The introduction

of the quantities $D_x\mu$, $D_y\mu$, and $D_z\mu$ facilitates the transformation of surface integrals.

To conclude this section we present without proof several formulas of which we shall make partial use in the sequel.

We now assume in addition that the function F in the equations of the surface (11) possesses continuous second derivatives. The surface (S) then has a continuous curvature, and the formulas

$$\left.\begin{aligned} D_x \cos(Nx) &= \frac{\cos^2(L_1 x)}{R_1} + \frac{\cos^2(L_2 x)}{R_2} \\ D_x \cos(Ny) &= \frac{\cos(L_1 x)\cos(L_1 y)}{R_1} + \frac{\cos(L_2 x)\cos(L_2 y)}{R_2} \end{aligned}\right\} \qquad (27)$$

hold, wherein $\frac{1}{R_1}$ and $\frac{1}{R_2}$ are the negative principal curvatures and L_1 and L_2 are the corresponding principal directions of curvature on the surface (S).[3]

From formulas (27) and their analogues it follows that

$$D_x \cos(Ny) = D_y \cos(Nx),$$
$$D_x \cos(Nz) = D_z \cos(Nx),$$
$$D_y \cos(Nz) = D_z \cos(Ny).$$

We shall subsequently encounter the expression

$$D_x \cos(Nx) + D_y \cos(Ny) + D_z \cos(Nz),$$

which we denote by K. The first of formulas (27) implies the validity of the equation

$$K = \frac{1}{R_1} + \frac{1}{R_2};$$

K is hence the mean curvature of the surface S[4].

§4. A Finite Covering of a Surface

Let (S) be a closed LYAPUNOV surface. We shall show that it is possible to construct a finite number of LYAPUNOV spheres such that each point of (S) lies inside at least one of these spheres.

Let us consider a cube with edges parallel to the coordinate axes which contains the surface (S) in its interior. The length of an edge of this cube we denote by l. Let n be a natural number so large that the following inequality holds:

$$q = \frac{l}{n} < \frac{d}{\sqrt{3}}.$$

[3] The proof of formulas (27) is given in the Appendix to Chapter I.

[4] K is minus two times the mean curvature as usually defined (Trans.).

We subdivide the cube into n^3 cubes with edge length q. A certain number Q of these cubes will have points in common with (S), and Q is at most n^3. If a point M_0 of the surface (S) lies in the interior or on the surface of a cube with edge length q, then the entire cube lies inside a LYAPUNOV sphere about M_0, since the length of the diagonal of the cube is less than d. Putting a sphere in this manner about each cube having a point in common with (S), one obtains Q LYAPUNOV spheres having the desired property.

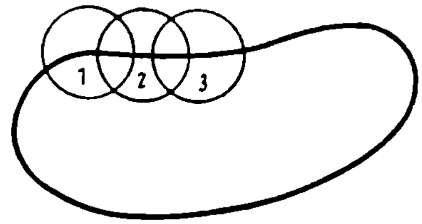

Fig. 6

If the radius d of the LYAPUNOV sphere satisfies condition (4), then the equation for the portion of the surface lying inside the LYAPUNOV sphere, and hence also the equation for the portion of the surface inside the cube, possesses one of the forms (11). Moreover, with appropriate choice of the coordinate system the equation of the surface can be represented in each of the forms (11). We shall make use of this fact in deriving the integral formulas of OSTROGRADSKII and STOKES.

§5. The Formulas of OSTROGRADSKII and STOKES

Theorem 1. *If a finite region (D) is bounded by a finite number of closed* LYAPUNOV *surfaces and if the functions U, V, and W have continuous and bounded first derivatives in (D), then*

$$\int_{(D)}\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}\right)d\tau=\int_{(S)}[U\cos(Nx)+V\cos(Ny)+W\cos(Nz)]\,d\sigma, \qquad (28)$$

where N denotes the direction of the outer normal to the surface (S), $d\tau$ is the volume element, and $d\sigma$ is the surface element.

The proof of this theorem, which is named for OSTROGRADSKII[5], can be found in any analysis textbook. It is proved under the hypothesis that the region (D) can be decomposed into a finite number of subregions in such a manner that parallels to coordinate axes suitably chosen for each subregion intersect the boundary of the subregion in at most two points. The cubes of §4 may be used for the decomposition of the region. For a cube which

[5] The theorem is usually referred to as GAUSS' Theorem or the Divergence Theorem (Trans.).

lies entirely in the interior of (D) formula (28) can be proved directly without introducing new coordinates. In subregions next to the boundary we choose the coordinate system such that the section of the surface (S) in question is cut at most once by parallels to the axes. The entire boundary of such a subregion will then be cut by these parallels at most twice.

Since the functions U, V, and W are arbitrary, it follows from (28) that

$$\left.\begin{aligned} \int\limits_{(D)} \frac{\partial U}{\partial x}\, d\tau &= \int\limits_{(S)} U \cos(Nx)\, d\sigma\,, \\ \int\limits_{(D)} \frac{\partial V}{\partial y}\, d\tau &= \int\limits_{(S)} V \cos(Ny)\, d\sigma\,, \\ \int\limits_{(D)} \frac{\partial W}{\partial y}\, d\tau &= \int\limits_{(S)} W \cos(Nz)\, d\sigma\,. \end{aligned}\right\} \tag{29}$$

Remark. It is not necessary to assume that $\frac{\partial U}{\partial x}$, $\frac{\partial V}{\partial y}$, and $\frac{\partial W}{\partial z}$ are continuous; it suffices to assume that these quantities are bounded and integrable. Since we shall make no use of this generalization, it is sufficient for our purposes simply to indicate the possibility of such a generalization.

Let us now consider the integral formula of STOKES.

Suppose that we are given three functions φ, ψ, and χ on the surface (S) which are differentiable (in the sense of §3). We consider on the surface (S) a certain subregion (σ) bounded by a curve (l).

Theorem 2. *Under the conditions given above the formula*

$$\begin{aligned} \int\limits_{(\sigma)} [(D_y\chi - D_z\psi)\cos(Nx) + (D_z\varphi - D_x\chi)\cos(Ny) + (D_z\psi - D_y\varphi)\cos(Nz)]\, d\sigma \\ = \int\limits_{(l)} (\varphi\, dx + \psi\, dy + \chi\, dz)\,, \end{aligned} \tag{30}$$

holds, where the boundary curve is oriented in such a manner that an observer standing upright in the direction of the normal N while traversing the curve (l) *has the region* (σ) *to his left.* (It is assumed that the coordinate system in rectangular.)

Proof. Suppose first of all that (σ) has been decomposed into subregions for each of which formula (30) has been proved. Adding the left sides of the corresponding formulas, we obtain the integral over (σ); on adding the right sides the integrals over the auxiliary contours drop out, since integrals over the same section of the auxiliary contour taken in opposite directions appear pairwise in the sum; therefore, only the integral over the boundary curve (l) remains. It therefore suffices to prove formula (30) for a section of (S) inside one of the Q spheres of §4.

The hypothesis of the existence, for example, of the quantity $D_y\varphi$ includes that of the existence of a function defined in a region bounded by (S) and

a portion of the LYAPUNOV sphere which assumes the value φ on (S). In order to avoid introducing new notation, we denote this function also by φ. We make a similar agreement for ψ and χ. We then have

$$\cos(Nx)\,D_y\chi - \cos(Ny)\,D_x\chi = \left(\frac{\partial\chi}{\partial y}\right)\cos(Nx) - \left(\frac{\partial\chi}{\partial x}\right)\cos(Ny)\,,$$

and formula (30) assumes the following form:

$$\iint\limits_{(\sigma)}\left\{\left[\left(\frac{\partial\chi}{\partial y}\right)-\left(\frac{\partial\psi}{\partial z}\right)\right]\cos(Nx)+\left[\left(\frac{\partial\varphi}{\partial z}\right)-\left(\frac{\partial\chi}{\partial x}\right)\right]\cos(Ny)\right.$$
$$\left.+\left(\frac{\partial\psi}{\partial x}\right)-\left(\frac{\partial\varphi}{\partial y}\right)\right]\cos(Nz)\right\}d\sigma=\int\limits_{(l)}(\varphi\,dx+\psi\,dy+\chi\,dz)\,. \quad (31)$$

The proof of this formula for a continuously deformed surface can be found in any textbook of analysis. For LYAPUNOV surfaces it must be proved with certain precautions.

We consider a vector field which in a particular coordinate system has the coordinates φ, ψ, and χ. On transforming to another rectangular coordinate system the new coordinates of the vector field are expressed as a linear combination of φ, ψ, and χ. It is well known that the integrands of both the left and right sides of formula (31) hereby retain their form, i.e., φ, ψ, and χ are simply replaced by the new coordinates of the field and x, y, and z by the new point coordinates. It is therefore sufficient to establish the validity of formula (31) in an arbitrary coordinate system.

To avoid introducing further notation we shall assume in the proof of formula (31) that φ, ψ, and χ are the new coordinates of the vector field and x, y, and z are the new point coordinates in the system under consideration.

We recall that if the radius d of the LYAPUNOV sphere satisfies condition (4) then one can choose coordinate axes in each LYAPUNOV sphere such that the portion of (S) inside the sphere is not cut in more than one point by parallels to any of the axes nor are any such parallels tangent to this portion of (S). We assume that in the sphere under consideration the axes have been thus chosen. In this case the surface may be represented in any one of the three forms:

$$z = F(x,y)\,, \quad x = F(y,z)\,, \quad y = F(z,x)\,. \quad (11')$$

The proof of formula (31) reduces to proving the following three identities:

$$\left.\begin{aligned}
&\int\limits_{(\sigma)}\left[\left(\frac{\partial\varphi}{\partial z}\right)\cos(Ny)-\left(\frac{\partial\varphi}{\partial y}\right)\cos(Nz)\right]d\sigma=\int\limits_{(l)}\varphi\,dx\,,\\
&\int\limits_{(\sigma)}\left[\left(\frac{\partial\psi}{\partial x}\right)\cos(Nz)-\left(\frac{\partial\psi}{\partial z}\right)\cos(Nx)\right]d\sigma=\int\limits_{(l)}\psi\,dy\,,\\
&\int\limits_{(\sigma)}\left[\left(\frac{\partial\chi}{\partial y}\right)\cos(Nx)-\left(\frac{\partial\chi}{\partial x}\right)\cos(Ny)\right]d\sigma=\int\limits_{(l)}\chi\,dz\,.
\end{aligned}\right\}\qquad(32)$$

To prove the first of these equations we consider the first of the three equations (11'):

$$z=F(x,y)\,.$$

Recalling the relations

$$\frac{\partial z}{\partial x}=-\frac{\cos(Nx)}{\cos(Nz)}\,,\qquad\frac{\partial z}{\partial y}=-\frac{\cos(Ny)}{\cos(Nz)}\,,$$

we obtain

$$\begin{aligned}
\left(\frac{\partial\varphi}{\partial z}\right)\cos(Ny)-\left(\frac{\partial\varphi}{\partial y}\right)\cos(Nz)&=-\left[\left(\frac{\partial\varphi}{\partial z}\right)\frac{\partial z}{\partial y}+\left(\frac{\partial\varphi}{\partial y}\right)\right]\cos(Nz)\\
&=-\frac{\partial(\varphi)}{\partial y}\cos(Nz)\,,
\end{aligned}$$

where (φ) is the expression arising when z in $\varphi(x,y,z)$ is replaced by $F(x,y)$. From this it follows that the integral over (σ) assumes the value

$$-\int\limits_{(\sigma)}\frac{\partial(\varphi)}{\partial y}\cos(Nz)\,d\sigma=-\eta\int\limits_{(\bar\sigma)}\frac{\partial(\varphi)}{\partial y}\,dx\,dy\,,\qquad(\eta=\pm1)$$

where $\eta=1$ if $\cos(Nz)>0$, and $\eta=-1$ if $\cos(Nz)<0$; $(\bar\sigma)$ denotes the projection of σ on the xy plane. Now

$$-\eta\int\limits_{(\bar\sigma)}\frac{\partial(\varphi)}{\partial y}\,dx\,dy=\eta\int\limits_{(\bar l)}(\varphi)\,dx\,,$$

where $(\bar l)$ denotes the boundary of $(\bar\sigma)$ and is traversed in such a manner that an observer upright in the direction of the positive z axis hereby has the region $(\bar\sigma)$ to his left. If $\cos(Nz)>0$, then $\eta=1$, and an observer standing upright on (S) in the direction of the normal to the surface (σ) finds that the integration proceeds in such a way that the surface (σ) remains to his left; in this case when passing from $(\bar l)$ to (l) the formula need not be altered (Fig. 7, right surface). If $\cos(Nz)<0$, then $\eta=-1$, and the observer finds the integration proceeding such that the surface (σ) remains on his right; in this case it is necessary to change the sign of the integral when passing from $(\bar l)$ to (l) (Fig. 7, left surface).

If the surface is represented by the second or third of the equations (11'),

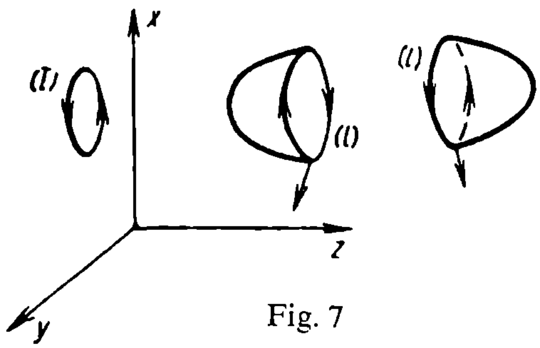

Fig. 7

then by similar considerations one obtains the second or third of the identities (32). Formula (31) is herewith proved and hence also formula (30) which follows from it.

§6. A Remark on the Integration of Unbounded Functions

In this section we wish to establish the definition of the integral of an unbounded function over a finite three-dimensional region or over a finite surface.

To have a specific case at hand, we consider a three-dimensional region (D). Let (Σ) be a surface, the points of which lie either in the interior of (D) or on the boundary. Let $F(x,y,z) = F(M)$ be a function which is continuous at every point of (D) not belonging to (Σ) and may become unbounded in a neighborhood of any point of (Σ). Further, let (ω) be an arbitrary region containing the surface (Σ) in its interior which possesses the property that the distance of any one of its points from (Σ) is not greater than a certain number $\delta > 0$. In the region $(D - \omega)$, which consists of points of (D) not belonging to (ω), the function $F(M)$ is continuous and bounded; the integral

$$\int\limits_{(D-\omega)} F(M)\, d\tau \tag{33}$$

therefore exists. We say that the integral (33) tends to a limit a as $\delta \to 0$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\left| \int\limits_{(D-\omega)} F(M)\, d\tau - a \right| < \varepsilon,$$

whenever the distance of each of the points of the region (ω) to the enclosed surface (Σ) is less than δ.

If as $\delta \to 0$ the integral (33) possesses a finite limit, then this limit is called the integral of $F(M)$ over the region (D), and one writes

$$\int\limits_{(D)} F(M)\, d\tau; \tag{34}$$

one says that the integral (34) is convergent.

Since subsequently it will be a question mainly of integrals of functions which are unbounded in the neighborhood of a single point, we shall henceforth consider only such functions. All results carry over in an obvious manner to the general case in which the singular points form a surface (Σ).

Let $(D(\delta_1,\delta_2))$ be the subregion of (D) enclosed between spherical surfaces of radii δ_1 and δ_2 $(0 < \delta_2 < \delta_1)$.

Theorem. *If the integral*

$$\int_{(D(\delta_1,\delta_2))} |F|\, d\tau$$

becomes arbitrarily small as $\delta_1 \to 0$, *then the integral* (34) *is convergent.*

Proof. We assume that (ω) and (ω') are two arbitrary regions which contain the sphere of radius δ_2 about M_0 and are themselves contained in the sphere of radius δ_1 about M_0 $(\delta_2 < \delta_1)$ (Fig. 8). The regions $(D - \omega)$ and $(D - \omega')$ then have the property that they do not contain the portion of (D) inside the sphere of radius δ_2, but have a common region which includes the portion of (D) lying outside the sphere of radius δ_1. Hence

$$\begin{aligned}\left| \int_{(D-\omega)} F\, d\tau - \int_{(D-\omega')} F\, d\tau \right| &= \left| \int_{(D(\delta_1,\delta_2)-\omega)} F\, d\tau - \int_{(D(\delta_1,\delta_2)-\omega')} F\, d\tau \right| \\ &\le \left| \int_{(D(\delta_1,\delta_2)-\omega)} F\, d\tau \right| + \left| \int_{(D(\delta_1,\delta_2)-\omega')} F\, d\tau \right| \\ &\le \int_{(D(\delta_1,\delta_2)-\omega)} |F|\, d\tau + \int_{(D(\delta_1,\delta_2)-\omega')} |F|\, d\tau < 2\int_{(D(\delta_1,\delta_2))} |F|\, d\tau ,\end{aligned}$$

i.e., the difference

$$\int_{(D-\omega)} F\, d\tau - \int_{(D-\omega')} F\, d\tau$$

becomes arbitrarily small as $\delta_1 \to 0$; according to the CAUCHY criterion this ensures the existence of a limit for (33) as $\delta_1 \to 0$.

If the integral

$$\int_{(D)} |F|\, d\tau$$

converges, one says that the integral (34) is absolutely convergent. From the theorem just proved it follows immediately that absolute convergence of the integral implies convergence. Without going into the proof, let us note that with our definition of convergence the convergence of the integral also implies absolute convergence.

We similarly define the integral $\int_{(S)} F\, d\sigma$ of a function F over a surface (S) when the function is continuous at all points of the surface with the exception of the point M_0.

In place of the spheres which we used in investigating integrals over (D)

we here use subregions of (S) cut out by circular cylinders with the normal at M_0 as axis.

Example. Let M_0 be a fixed point of the surface (S), M a variable point of this surface, and r_{10} the distance between M_0 and M. Further let N be the normal to (S) at the point M and $(r_{10}N)$ the angle between the directions of r_{10} and N (Fig. 9). Finally, let μ be a bounded and integrable function of M. We form the integral

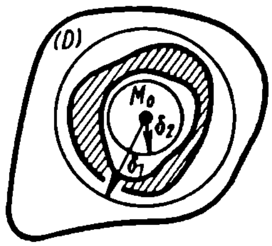

Fig. 8

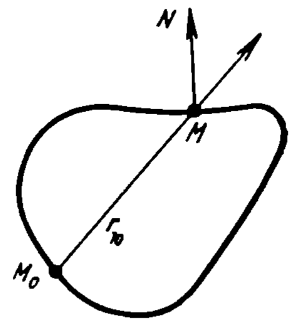

Fig. 9

$$\int\limits_{(S)} \mu \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma . \tag{35}$$

This integral, as a function of a point M_0 whether on (S) or not, is called the *potential of a double layer*; the function μ is called the density of the layer.

The integrand in (35) becomes infinite as M tends to the point M_0 on (S). We wish to show that the integral (35) is convergent. Let (σ_1) and (σ_2) be subregions of (S) lying inside a LYAPUNOV sphere about M_0 which are cut out by circular cylinders with radii δ_1 and δ_2 $(\delta_2 < \delta_1)$ and the normal at M_0 as axis (Fig. 10). By the theorem proved, to establish convergence of the integral (35) it is sufficient to show that the integral

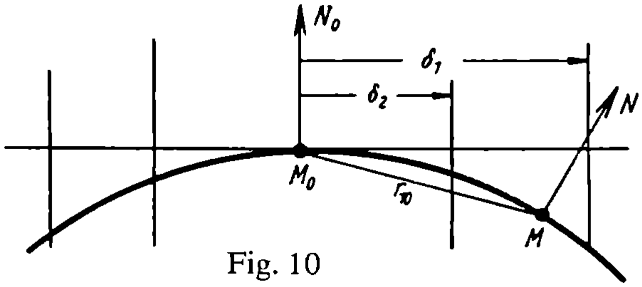

Fig. 10

$$\int\limits_{(\sigma_1-\sigma_2)} |\mu| \frac{|\cos(r_{10}N)|}{r_{10}^2} d\sigma \tag{36}$$

becomes arbitrarily small as $\delta_1 \to 0$. From estimate (17) of §1 we have:

$$|\cos(r_{10}N)| < E r_{10}^{\lambda}.$$

If we denote by A the upper bound of $|\mu|$ on (S), then making use of estimates (16) and (12) of §1 we obtain:

$$\int_{(\sigma_1-\sigma_2)} |\mu| \frac{|\cos(r_{10}N)|}{r_{10}^2} d\sigma < A \cdot 2E \int_{(\sigma_1^{(0)}-\sigma_2^{(0)})} \frac{r_{10}^{\lambda}}{r_{10}^2} d\sigma^{(0)}$$

$$= 2EA \int_{(\sigma_1^{(0)}-\sigma_2^{(0)})} \frac{d\sigma^{(0)}}{r_{10}^{2-\lambda}} \leq 2EA \int_0^{2\pi} \int_{\delta_2}^{\delta_1} \frac{\varrho\, d\varrho\, d\varphi}{\varrho^{2-\lambda}}$$

$$= 4\pi EA \frac{\delta_1^{\lambda} - \delta_2^{\lambda}}{\lambda} < \frac{4\pi E}{\lambda} A \delta_1^{\lambda},$$

i.e.,

$$\int_{(\sigma_1-\sigma_2)} |\mu| \frac{|\cos(r_{10}N)|}{r_{10}^2} d\sigma < cA\delta_1^{\lambda}; \tag{37}$$

hence

$$\int_{(\sigma)} |\mu| \frac{|\cos(r_{10}N)|}{r_{10}^2} d\sigma < cA\delta^{\lambda}, \tag{38}$$

where (σ) is the subregion of (S) cut out by the cylinder of radius δ. We shall make frequent use of inequality (38) in the second chapter.

Remark. From the preceding considerations it follows that

$$\left| \int_{(S)} \mu \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma \right| < aA,$$

where A is an upper bound for $|\mu|$. If μ_n is a sequence which converges uniformly to the limit function μ, it then follows that

$$\lim_{n\to\infty} \int_{(S)} \mu_n \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma = \int_{(S)} \mu \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma. \tag{39}$$

Indeed, by choice of an appropriate n_0 we obtain for $n \geqq n_0$:

$$\left| \int_{(S)} \mu_n \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma - \int_{(S)} \mu \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma \right| = \left| \int_{(S)} (\mu_n - \mu) \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma \right| < a\varepsilon.$$

A remark on the interchange of the order of integration. Let (D) be a finite region of space and (S) a surface of finite area. Let us denote points of the region (D) by M_1 and points of the surface (S) by M_2. Let $F(1, 2)$ be a function of M_1 and M_2 with the property that $F(M_1,M_2)$ is continuous as a function of M_1 in (D) for each fixed point M_2 with the exception of

points of a certain surface (or of a certain curve or certain isolated points) which depend on the choice of M_2. The integral

$$\int_{(D)} F(1,2)\, d\tau$$

is, when it exists, a possibly unbounded function of M_2. If this function is integrable on (S), then the integral

$$\int_{(S)} \left[\int_{(D)} F(M_1, M_2)\, d\tau_1 \right] d\sigma_2 \tag{40}$$

has meaning.

We assume further that $F(M_1,M_2)$ is continuous on (S) as a function of M_2 for each fixed point M_1 of (D) with the exception of points on a certain curve (or certain isolated points) which depend on the choice of M_1. The integral of $F(M_1,M_2)$ over (S) is, when it exists, a function of M_1. It may happen that the integral

$$\int_{(D)} \left[\int_{(S)} F(M_1, M_2)\, d\sigma_2 \right] d\tau_1 \tag{41}$$

then also exists.

The integrals (40) and (41) may turn out to be equal. It is possible, however, that only one of the integrals is meaningful, or that while both integrals have meaning they possess different values.

The theory of the multiple Lebesgue integral makes it possible to give a sufficient condition for the equality of the integrals (40) and (41). If one of the two integrals

$$\int_{(S)} \left[\int_{(D)} |F(1,2)|\, d\tau_1 \right] d\sigma_2 \tag{40'}$$

and

$$\int_{(D)} \left[\int_{(S)} |F(1,2)|\, d\sigma_2 \right] d\tau_1 \tag{41'}$$

exists, then the other also exists, and moreover the integrals (40) and (41) exist and are equal.

An analogous theorem applies when both integrals are defined on a surface or extend over an element of volume.

Let $F(0,1,2)$ be a function of points M_0, M_1, and M_2 which may, for example, lie on a surface (S). We consider the integral

$$\int_{(S_2)} \left\{ \int_{(S_1)} \left[\int_{(S)} F(0,1,2)\, d\sigma \right] d\sigma_1 \right\} d\sigma_2 .$$

This integral exists and is equal to any of the integrals obtained by interchanging the order of integration if the integral

$$\int\limits_{(S_2)}\left\{\int\limits_{(S_1)}\left[\int\limits_{(S)}|F(0,1,2)|\,d\sigma\right]d\sigma_1\right\}d\sigma_2$$

itself exists, or if one of the integrals obtained from this one by interchanging the order of integration exists.

If a function $F(1,2)$ is continuous so long as M_1 and M_2 are distinct, and if it satisfies the inequality

$$|F(1,2)| < \frac{C}{r^\alpha},$$

where C is a constant, r the distance between the points M_1 and M_2, and α is a constant such that $0 < \alpha < 3$, then one says that $F(1,2)$ possesses a polar singularity. For such functions the equality of the integrals (40) and (41) can be proved quite easily without using the theory of the LEBESGUE integral.[6] The unbounded functions which subsequently occur will for the most part possess polar singularities.

If a function $F(1,2)$ of points M_1 and M_2 of a surface (S) has a polar singularity, then for equality of the integrals

$$\int\limits_{(S_1)}\left[\int\limits_{(S_2)}F(1,2)\,d\sigma_2\right]d\sigma_1 \qquad \text{and} \qquad \int\limits_{(S_2)}\left[\int\limits_{(S_1)}F(1,2)\,d\sigma_1\right]d\sigma_2$$

it is sufficient that $\alpha < 2$.

§7. On Harmonic Functions

We begin with the following definitions:

a) In a region (D) not containing the infinitely distant point a function V is called *harmonic* if it satisfies the following three conditions:

1. V is defined in (D) and possesses uniformly continuous first derivatives in the interior of (D).

2. In the interior of (D) V possesses second derivatives which are continuous in every region (D_1) which together with its boundary is contained in (D).

3. At each interior point of (D) the second derivatives of V satisfy the LAPLACE *equation*

$$\Delta V \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{42}$$

[6] V. I. SMIRNOV, *Kurs vysshei matematiki*, Vol. IV, 2nd Ed., Moscow-Leningrad, 1951, pp. 66-68.

a′) A function V is called harmonic in the interior of (D) if it is harmonic in every region (D_1) which together with its boundary is contained in (D) and if it is uniformly continuous in (D). We shall subsequently see that the function

$$\int_{(S)} \frac{\mu\, d\sigma}{r_{10}}$$

is harmonic in (D) if μ is H-continuous on (S), while under the same conditions on the density μ the function (35) is harmonic only in the interior of (D), in general.

b) In a region (D_e) lying outside of (D) and containing the infinitely distant point a function V is called harmonic if it satisfies the following four conditions:

1. The function is defined in (D_e), and its first derivatives are uniformly continuous in the interior of (D_e).

2. Its second derivatives are continuous in every region (D_e') which together with its boundary is contained in (D_e).

3. At every interior point of (D_e) the second derivatives of V satisfy the LAPLACE equation.

4. If R is the distance of a point of the region (D_e) to a certain fixed point, then

$$|RV| < A, \qquad \left|R^2 \frac{\partial V}{\partial x}\right| < A, \qquad \left|R^2 \frac{\partial V}{\partial y}\right| < A, \qquad \left|R^2 \frac{\partial V}{\partial z}\right| < A,$$

where A is a constant.[7]

b′) A function is called harmonic in the interior of (D_e) if it is harmonic in every region (D_e') containing the infinitely distant point and contained in (D_e) and is uniformly continuous in (D_e).

From the definitions given it follows that a nonzero constant C is harmonic in (D_i) but not in (D_e). We shall henceforth always denote by (D_i) the region which does not contain the infinitely distant point and by (D_e) the region which enlarges (D_i) to the entire space.

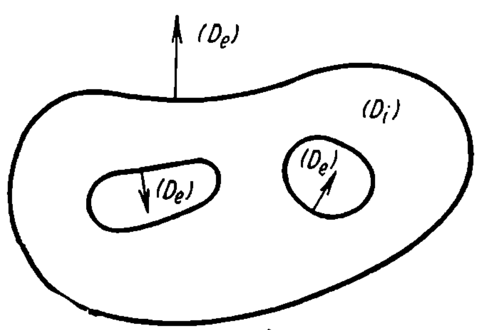

Fig. 11

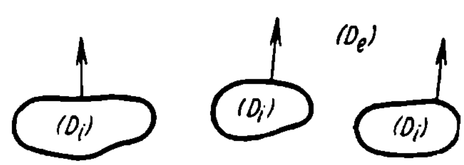

Fig. 12

[7] In IV, §1 it will be proved that every function satisfying conditions 1, 2, and 3 which tends to zero as $R \to \infty$ also satisfies the inequalities 4. One could therefore replace condition 4 by the requirement: $V \to 0$ for $R \to \infty$.

It is to be noted that we do not exclude either regions (D_i) with inner boundaries (Fig. 11) or regions (D_e) bounded by several surfaces (Fig. 12).

We shall agree that the normals of the boundary of a region (D_i) will always be directed into the interior of the corresponding region (D_e).

The region (D_e) corresponding to the region (D_i) of Figure 11 is not connected, likewise the region (D_i) corresponding to the region (D_e) of Figure 12.

§8. GREEN'S Identities

Let two harmonic functions U and V be given in a region (D_i); we form the integral
$$I = \int\limits_{(D_i)} \left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial V}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial V}{\partial z} \right) d\tau,$$
which for brevity we shall write in the form
$$\int\limits_{(D_i)} \sum \frac{\partial U}{\partial x}\frac{\partial V}{\partial x}\, d\tau.$$
Making use of the formula of OSTROGRADSKII, we obtain:
$$\begin{aligned} I &= \int\limits_{(D_i)} \left[\frac{\partial}{\partial x}\left(U\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(U\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(U\frac{\partial V}{\partial z}\right)\right] d\tau \\ &\quad - \int\limits_{(D_i)} U\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}\right) d\tau \\ &= \int\limits_{(D_i)} \left[\frac{\partial\left(U\frac{\partial V}{\partial x}\right)}{\partial x} + \frac{\partial\left(U\frac{\partial V}{\partial y}\right)}{\partial y} + \frac{\partial\left(U\frac{\partial V}{\partial z}\right)}{\partial z} \right] d\tau \\ &= \int\limits_{(S)} U\left[\frac{\partial V}{\partial x}\cos(Nx) + \frac{\partial V}{\partial y}\cos(Ny) + \frac{\partial V}{\partial z}\cos(Nz)\right] d\sigma = \int\limits_{(S)} U\frac{dV_i}{dn}\, d\sigma. \end{aligned}$$

Remark. If the boundary consists of several surfaces, the integral
$$\int\limits_{(S)} U\frac{dV_i}{dn}\, d\sigma$$
denotes the sum of the integrals over these surfaces.

We have hereby provided the normal derivative $\frac{dV}{dn}$ with the index i to indicate that in forming the derivative the boundary is approached from the interior of (D_i).

Interchanging the roles of the functions U and V in the preceding calculation, one obtains:

$$I = \int_{(S)} V \frac{dU_i}{dn} d\sigma .$$

Hence

$$\int_{(D)} \sum \frac{\partial U}{\partial x} \frac{\partial V}{\partial x} d\tau = \int_{(S)} U \frac{dV_i}{dn} d\sigma = \int_{(S)} V \frac{dU_i}{dn} d\sigma \tag{43}$$

and

$$\int_{(S)} \left(U \frac{dV_i}{dn} - V \frac{dU_i}{dn} \right) d\sigma = 0 . \tag{44}$$

Formulas (43) and (44) are called GREEN's *identities* for functions which are harmonic in (D_i). Since $V \equiv 1$ is a harmonic function in (D_i), it follows from (44) that

$$\int_{(S)} \frac{dU_i}{dn} d\sigma = 0 , \tag{45}$$

i.e., the integral over the boundary of the normal derivative of a function harmonic in (D_i) is equal to zero.

Before proceeding to analogous formulas for functions harmonic in (D_e), we define the integral over an infinite region.

If (D) is an infinite region we shall say that the sequence $\{(D_n)\}$ of finite regions (D_n) $(n = 1, 2, \ldots)$ exhausts the region (D) if (D_n) is contained in (D_{n+1}) and every point of (D) is contained in at least one region of the sequence.

Let F be a function continuous in (D). We form the integral

$$\int_{(D_n)} F d\tau .$$

If the limit of this integral for $n \to \infty$ exists and is independent of the choice of the sequence of regions, we call this limit the integral of F over (D) and write

$$\int_{(D)} F d\tau .$$

We now consider the integral

$$\int_{(D_e)} \sum \frac{\partial U}{\partial x} \frac{\partial V}{\partial x} d\tau ,$$

where U and V are harmonic functions in (D_e). It is noted without proof

that the existence of this integral is guaranteed by the inequalities in the fourth condition for functions harmonic in (D_e).

For functions harmonic in (D_e) GREEN'S identities have the form:

$$\int_{(D_e)} \sum \frac{\partial U}{\partial x} \frac{\partial V}{\partial x} d\tau = -\int_{(S)} U \frac{dV_e}{dn} d\sigma = -\int_{(S)} V \frac{dU_e}{dn} d\sigma, \tag{43'}$$

$$\int_{(S)} \left(U \frac{dV_e}{dn} - V \frac{dU_e}{dn} \right) d\sigma = 0. \tag{44'}$$

For the proof we consider the surface (Σ) of a sphere about a certain fixed point whose radius R we choose so large that the entire boundary of the region (D_e) lies inside this sphere. We apply formula (43) to the subregion (D_e') of (D_e) bounded by (Σ). It must be noted that in employing this formula the direction of the normals of (S) must be reversed, since relative to the region (D_e') the region (D_i) is this time a part of the outer region.

Choosing the direction of the normals in all formulas according to the previous agreement, we find

$$\int_{(D_e')} \sum \frac{\partial U}{\partial x} \frac{\partial V}{\partial x} d\tau = -\int_{(S)} U \frac{dV_e}{dn} d\sigma + \int_{(\Sigma)} U \frac{dV}{dn} d\sigma. \tag{46}$$

As R tends to infinity, the last integral tends to zero. Indeed, the functions U and V defined in (D_e) satisfy the fourth condition; hence

$$|U| < \frac{A}{R},$$

$$\left| \frac{\partial V}{\partial x} \cos(Nx) + \frac{\partial V}{\partial y} \cos(Ny) + \frac{\partial V}{\partial z} \cos(Nz) \right|$$
$$\leq \sqrt{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2} < \frac{\sqrt{3}A}{R^2}.$$

Introducing polar coordinates on (Σ),

$$x = R \sin \Theta \cos \varphi, \quad y = R \sin \Theta \sin \varphi, \quad z = R \cos \Theta$$
$$(0 \leq \varphi < 2\pi, \ 0 \leq \Theta \leq \pi)$$

we have

$$d\sigma = R^2 \sin \Theta \, d\Theta \, d\varphi,$$

and hence

$$\left| \int_{(\Sigma)} U \frac{dV}{dn} d\sigma \right| \leq \frac{\sqrt{3}A^2}{R^3} \int_0^{2\pi} \int_0^{\pi} R^2 \sin \Theta \, d\Theta \, d\varphi = \frac{\sqrt{3}A^2 4\pi}{R} \to 0 \quad (R \to \infty).$$

Moreover, as noted above the integral on the left side of (43′) exists and is in particular equal to the integral on the left side of (46) for $R \to \infty$. Formula (43′) and hence also (44′) follows from (46) for $R \to \infty$.

Let $M_0(x,y,z)$ be a fixed point and r_{10} its distance from a variable point $M_1(\xi,\eta,\zeta)$. We shall assume that the segment M_0M_1 from M_0 to M_1 is oriented, and we put

$$U(\xi, \eta, \zeta) = \frac{1}{r_{10}} = \frac{1}{\sqrt{(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2}} \tag{47}$$

The function U is continuous and possesses continuous derivatives of arbitrary order in every region which does not contain the point M_0 either in its interior or on the boundary. By direct calculation one finds that the equation

$$\frac{\partial^2 U}{\partial \xi^2} + \frac{\partial^2 U}{\partial \eta^2} + \frac{\partial^2 U}{\partial \zeta^2} = 0$$

holds, i.e., in any finite region which does not contain the point M_0 either in its interior or on the boundary U is a harmonic function. Since U clearly also satisfies the fourth condition, this holds also for an infinite region.

Let N be a unit vector and $\frac{dU}{dn}$ be the derivative of U in the direction N. In order to be able to apply formulas (44) and (44′) to U, we first compute $\frac{dU}{dn}$. Since

$$\cos(r_{10}x) = \frac{\xi - x}{r_{10}}, \qquad \cos(r_{10}y) = \frac{\eta - y}{r_{10}}, \qquad \cos(r_{10}z) = \frac{\zeta - z}{r_{10}}$$

we find

$$\begin{aligned} \frac{dU}{dn} &= \frac{\partial U}{\partial \xi}\cos(Nx) + \frac{\partial U}{\partial \eta}\cos(Ny) + \frac{\partial U}{\partial \zeta}\cos(Nz) \\ &= -\frac{1}{r_{10}^2}\left[\frac{\xi - x}{r_{10}}\cos(Nx) + \frac{\eta - y}{r_{10}}\cos(Ny) + \frac{\zeta - z}{r_{10}}\cos(Nz)\right] \\ &= -\frac{\cos(r_{10}N)}{r_{10}^2}. \end{aligned} \tag{48}$$

Returning to formula (44), let V be a harmonic function in (D_i). We first assume that $M_0(x,y,z)$ lies in the interior of (D_e). In this case, U is harmonic in (D_i), and formula (44) is applicable. Making use of (47) and (48), we obtain:

$$\int_{(S)} \frac{dV_i}{dn}\frac{d\sigma}{r_{10}} + \int_{(S)} V\frac{\cos(r_{10}N)}{r_{10}^2}\,d\sigma = 0. \tag{49}$$

We now assume that $M_0(x,y,z)$ lies in the interior of (D_i). U is then no longer continuous in (D_i), and formula (44) no longer applies. Let (σ) be

the surface of a sphere about M_0 whose radius ϱ is so small that there are no points of (S) in its interior or on (σ) (Fig. 13).

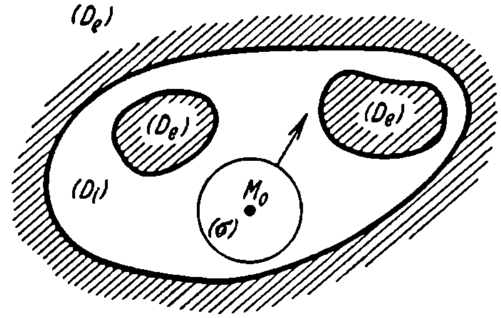

Fig. 13

In any region outside this sphere, and in particular in the subregion of (D_i) outside this sphere, U is a harmonic function. Applying formula (44) to this subregion, we obtain:

$$\int\limits_{(S)} \frac{dV_i}{dn}\frac{d\sigma}{r_{10}} + \int\limits_{(S)} V\frac{\cos(r_{10}N)}{r_{10}^2}\,d\sigma - \int\limits_{(\sigma)} \frac{dV}{dn}\frac{d\sigma}{r_{10}} - \int\limits_{(\sigma)} V\frac{\cos(r_{10}N)}{r_{10}}\,d\sigma = 0. \qquad (50)$$

We have here changed the sign of the integral over (σ), since in applying formula (44) the normal to (σ) must be directed into the interior of the sphere enclosed by (σ), while in formula (50) we assume that it is directed outward from the sphere, i.e., into the interior of (D_i).

We now compute the integral over (σ). First of all,

$$\left|\int\limits_{(\sigma)} \frac{dV_i}{dn}\frac{d\sigma}{r_{10}}\right| \leq \sqrt{3}\,A\int\limits_0^{2\pi}\int\limits_0^{\pi}\frac{\varrho^2\sin\Theta\,d\Theta\,d\varphi}{\varrho} = 4\pi\sqrt{3}A\varrho \to 0 \qquad (\varrho \to 0).$$

A is here an upper bound for the first derivatives of V in (D_i).

If V_0 is the value of V at the center of the sphere, then

$$|V - V_0| \leq \sqrt{3}A\varrho.$$

From this it follows that

$$\left|\int\limits_{(\sigma)}(V - V_0)\frac{\cos(r_{10}N)}{r_{10}^2}\,d\sigma\right|$$
$$\leq \sqrt{3}A\varrho\int\limits_0^{2\pi}\int\limits_0^{\pi}\frac{\varrho^2\sin\Theta\,d\Theta\,d\varphi}{\varrho^2} = 4\pi\sqrt{3}A\varrho \to 0 \qquad (\varrho \to 0),$$

and hence

$$\int_{(\sigma)} V \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma = V_0 \int_{(\sigma)} \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma$$
$$+ \int_{(\sigma)} (V - V_0) \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma \to 4\pi V_0 \qquad (\varrho \to 0).$$

Therefore, if M_0 lies in (D_i)

$$V_0 = \frac{1}{4\pi} \int_{(S)} \frac{dV_i}{dn} \frac{d\sigma}{r_{10}} + \frac{1}{4\pi} \int_{(S)} V \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma. \tag{51}$$

It is possible to extend formulas (49) and (51) to functions defined in the region (D_e).

If M_0 lies in the interior of (D_i), then U is continuous in (D_e), and application of formula (44′) gives immediately that

$$\int_{(S)} \frac{dV_e}{dn} \frac{d\sigma}{r_{10}} + \int_{(S)} V_e \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma = 0. \tag{51′}$$

Let M_0 now lie in the interior of (D_e). We consider the surface (Σ) of a sphere of radius R with center M_0 which contains the entire boundary of (D_e) and apply formula (51) to the region bounded by (Σ) and the boundary of (D_e). We hereby obtain:

$$\begin{aligned} V_0 = {} & \frac{1}{4\pi} \int_{(S)} \frac{dV_e}{dn} \frac{d\sigma}{r_{10}} + \frac{1}{4\pi} \int_{(\Sigma)} \frac{dV_e}{dn} \frac{d\sigma}{r_{10}} \\ & - \frac{1}{4\pi} \int_{(S)} V_e \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma + \frac{1}{4\pi} \int_{(\Sigma)} V \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma. \end{aligned} \tag{52}$$

The sign of the integrals over (S) has been changed, since in applying formula (51) to the region under consideration the normals must be directed into the interior of (D_i), while we have them directed into the interior of (D_e).

Now because of condition 4 we find

$$\left.\begin{aligned} \left| \int_{(\Sigma)} \frac{dV_e}{dn} \frac{d\sigma}{r_{10}} \right| &< \frac{\sqrt{3}A}{R^2} \int_0^{2\pi} \int_0^{\pi} \frac{R^2 \sin\Theta \, d\Theta \, d\varphi}{R} = \frac{4\pi\sqrt{3}A}{R} \to 0, \\ \left| \int_{(\Sigma)} V \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma \right| &< \frac{A}{R} \int_0^{2\pi} \int_0^{\pi} \frac{R^2 \sin\Theta \, d\Theta \, d\varphi}{R^2} = \frac{4\pi A}{R} \to 0. \end{aligned}\right\} (R \to \infty).$$

From this it follows that the integrals over (Σ) have limit zero as $R \to \infty$.

Let us now compare the various formulas found in this section. If M_0 lies in the interior of (D_e), then

$$\int\limits_{(S)} \frac{dV_i}{dn} \frac{d\sigma}{r_{10}} + \int\limits_{(S)} V_i \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma = 0; \tag{49}$$

if M_0 lies in the interior of (D_i), then

$$V = \frac{1}{4\pi} \int\limits_{(S)} \frac{dV_i}{dn} \frac{d\sigma}{r_{10}} + \frac{1}{4\pi} \int\limits_{(S)} V_i \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma; \tag{51}$$

for M_0 in (D_e), we have

$$V = -\frac{1}{4\pi} \int\limits_{(S)} \frac{dV_e}{dn} \frac{d\sigma}{r_{10}} - \frac{1}{4\pi} \int\limits_{(S)} V_e \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma; \tag{49'}$$

and for M_0 in (D_i),

$$\int\limits_{(S)} \frac{dV_e}{dn} \frac{d\sigma}{r_{10}} + \int\limits_{(S)} V_e \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma = 0. \tag{51'}$$

We now make use of these formulas to prove an important property of harmonic functions:

Theorem. *A function harmonic in any arbitrary region cannot assume a maximum or minimum at an interior point of the region.*

Proof. We prove the theorem for the case of a maximum. There is no loss of generality in so doing, since if a function U assumes a minimum at a point, then the function $-U$ assumes a maximum at this point.

We assume that the function U has a maximum at a point M_0 in the interior of the particular region; we do not hereby exclude the case in which U is constant in a region containing the point M_0. If this should be the case, we assume that M_0 is a boundary point of this region.

Let (σ) be the surface of a sphere about M_0 whose radius ϱ is chosen so small that both the sphere and its surface are contained in the region under consideration.

Applying formula (51) to the function U and the surface (σ), we obtain:

$$\begin{aligned} U_{M_0} &= \frac{1}{4\pi} \int\limits_{(\sigma)} U \frac{\cos(r_{10}N)}{r_{10}^2} d\sigma + \frac{1}{4\pi} \int\limits_{(\sigma)} \frac{dU_i}{dn} \frac{d\sigma}{r_{10}} \\ &= \frac{1}{4\pi\varrho^2} \int\limits_{(\sigma)} U\, d\sigma + \frac{1}{4\pi\varrho} \int\limits_{(\sigma)} \frac{dU_i}{dn} d\sigma, \end{aligned} \tag{51''}$$

since for the points of (σ), we have

$$r_{10} = \varrho, \quad \cos(r_{10}N) = 1.$$

The second integral in (51″) is zero by (45), and we find:

$$U_{M_0} = \frac{1}{4\pi\varrho^2}\int\limits_{(\sigma)} U\,d\sigma\,. \tag{53}$$

Putting

$$U = U_{M_0} - \alpha\,,$$

we see that α is a continuous, non-negative function, and equation (53) implies that

$$U_{M_0} = U_{M_0} - \frac{1}{4\pi\varrho^2}\int\limits_{(\sigma)} \alpha\,d\sigma \qquad \text{or} \qquad \int\limits_{(\sigma)} \alpha\,d\sigma = 0\,.$$

This is possible only if α vanishes identically on (σ). Since ϱ is arbitrary, it follows that α must also vanish identically in the interior of the sphere enclosed by (σ), however, this is impossible, since in a region in which U is constant M_0 by hypothesis lies on the boundary.

The assumption that U assumes a maximum at an interior point of the region has thus led to a contradiction, and the theorem is herewith proved.

Corollary. *A function harmonic in the interior of* (D_i) *assumes its maximum and its minimum on the boundary of this region.*

Proof. A function harmonic in (D_i) is continuous in this region including the boundary (S). Hence it assumes its maximum and minimum at certain points of the region or the boundary. From the theorem these points must lie on the boundary.

For functions harmonic in (D_e), the same assertion holds if the infinitely distant point is counted as a boundary point.

The property of harmonic functions just established is called the *Maximum Principle* for harmonic functions.

Finally, we consider the so-called *mean value theorems* for harmonic functions obtained from (53).

We assume in these theorems that the sphere (ω) and its boundary (σ) lie in a region in which U is harmonic.

Theorem 1. *The value of a harmonic function at the center of a sphere is equal to the mean value of its values on the surface of the sphere.*

This theorem follows immediately from formula (53), since $4\pi\,\varrho^2$ is the area of the surface of the sphere.

Theorem 2. *The value of a harmonic function at the center of a sphere is equal to the mean of its values throughout the sphere.*

Proof. Let R be the radius of the sphere (ω). Multiplying (53) by $4\pi\,\varrho^2\,d\varrho$ and integrating with respect to ϱ from 0 to R, we obtain after dividing by $\frac{4\pi\,R^3}{3} = \omega$:

$$U_{M_0} = \frac{1}{\omega}\int\limits_{(\omega)} U\,d\tau\,.$$

§9. GAUSS' Integral

We have shown in §7 that the function

$$V \equiv 1$$

is harmonic in (D_i). Putting $V \equiv 1$ in (49) and (51), we find:

$$\int_{(S)} \frac{\cos(r_{10} N)}{r_{10}^2} d\sigma = 0, \qquad \text{for } M \text{ in } (D_e). \tag{54}$$

$$\int_{(S)} \frac{\cos(r_{10} N)}{r_{10}^2} d\sigma = 4\pi, \qquad \text{for } M \text{ in } (D_i) \tag{54'}$$

The integral

$$W = \int_{(S)} \frac{\cos(r_{10} N)}{r_{10}^2} d\sigma \tag{55}$$

is called GAUSS' *integral.* It was shown in §6 that GAUSS' integral converges when the point M lies on the boundary (S). We shall now determine the value of GAUSS' integral for this case, under the condition of course that (S) is a LYAPUNOV surface. We assume that M is a boundary point and let (Σ) be the surface of a sphere of radius ϱ about M which cuts out the sub-region (σ) from (S). We denote by W_σ the GAUSS integral extended over the portion of (S) outside (σ):

$$W_\sigma = \int_{(S-\sigma)} \frac{\cos(r_{10} N)}{r_{10}^2} d\sigma .$$

By definition:

$$W = \lim_{\varrho \to 0} W_\sigma .$$

We denote by (Σ_1) the portion of (Σ) lying in the interior of (D_i) and by (Σ_2) the portion of (Σ) in the interior of (D_e) (Fig. 14). The integral (55) over $(S - \sigma)$ and (Σ_1) is zero, since the point M lies outside the region bounded by $(S - \sigma)$ and (Σ_1):

$$W_\sigma - \int_{(\Sigma_1)} \frac{\cos(r_{10} N)}{r_{10}^2} d\sigma = 0 . \tag{56}$$

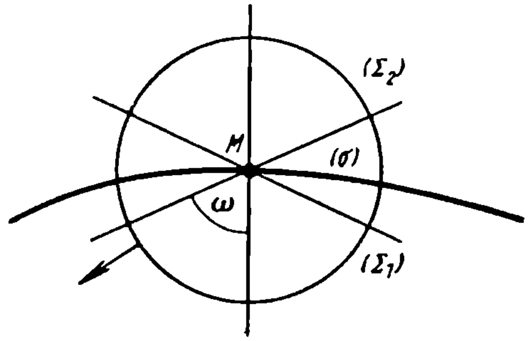

Fig. 14

We assume that N is the outer normal of (Σ). To apply formula (54), on the other hand, N must be the inner normal of (Σ). The integral in the last formula therefore has a negative sign.

If we now extend the integral (55) over $(S - \sigma)$ and (Σ_2) it has the value 4π since the point M now lies in the interior of the region bounded by $(S - \sigma)$ and (Σ_2); hence,

$$W_\sigma + \int\limits_{(\Sigma_2)} \frac{\cos(r_{10} N)}{r_{10}^2}\, d\sigma = 4\pi . \tag{56'}$$

Adding (56) and (56'), we find:

$$W_\sigma = 2\pi + \frac{1}{2}\left\{ \int\limits_{(\Sigma_1)} \frac{\cos(r_{10} N)}{r_{10}^2}\, d\sigma - \int\limits_{(\Sigma_2)} \frac{\cos(r_{10} N)}{r_{10}^2}\, d\sigma \right\}. \tag{57}$$

We shall now show that the quantity inside the braces in (57) becomes arbitrarily small as $\varrho \to 0$.

For this purpose we consider the (2ω)-cone corresponding to the point M. The integrals over the portions of (Σ_1) and (Σ_2) inside the cone cancel. Hence, the absolute value of the quantity in braces is less than the integral

$$\int\limits_{(\Sigma_3)} \frac{|\cos(r_{10} N)|}{r_{10}^2}\, d\sigma , \tag{58}$$

where (Σ_3) is the portion of (Σ) outside the cone.

After introducing polar coordinates, we obtain for this integral:

$$\int\limits_0^{2\pi} \int\limits_\omega^{\pi-\omega} \sin\Theta\, d\Theta\, d\varphi = -2\pi\left[\cos\Theta\right]_\omega^{\pi-\omega} = 4\pi\cos\omega .$$

Since we may put

$$\cos\omega = \frac{E\varrho^\lambda}{\sqrt{1 + \frac{1}{4} E^4 \varrho^{4\lambda}}}$$

it is evident that the limit of $\cos\omega$ is zero as $\varrho \to 0$.

We have therefore:

$$W = \lim_{\varrho \to 0} W_\sigma = 2\pi. \tag{59}$$

Altogether then, we have found the following:

$$\left.\begin{aligned} &W = 4\pi \text{ if } M \text{ lies in the interior of } (D_i),\\ &W = 2\pi \text{ if } M \text{ lies on the boundary of } (D_i), \text{ and}\\ &W = 0 \quad \text{if } M \text{ lies outside } (D_i). \end{aligned}\right\} \tag{60}$$

We remark that in deriving formulas (60), which are named for GAUSS', we have not assumed that the region (D_i) is bounded by a single surface.

It is useful to verify that formulas (60) are also correct in more general cases.

We first consider the case shown in Figure 12. If $(S_1), (S_2), \ldots, (S_k)$ are the boundaries of the separate connected regions, we obtain:

$$\int_{(S)} \frac{\cos(r_{10} N)}{r_{10}^2}\, d\sigma = \sum_{i=1}^{k} \int_{(S_i)} \frac{\cos(r_{10} N)}{r_{10}^2}\, d\sigma.$$

If M lies in the interior of (D_e) each of the integrals on the right-hand side of the equation is zero. If M lies in the region bounded by (S_i), then it lies outside the regions bounded by

$$(S_1), \ldots, (S_{i-1}). (S_{i+1}), \ldots, (S_k)\,,$$

and all integrals on the right-hand side of the equation vanish with the exception of the integral over (S_i); this integral is equal to 4π. If M lies on (S_i) we come to an analogous conclusion.

In the case shown in Figure 11 we denote the outer surface by (S_0) and the inner surfaces by $(S_1), \ldots, (S_k)$. In this case we have:

$$\int_{(S)} \frac{\cos(r_{10} N)}{r_{10}^2}\, d\sigma = \int_{(S_0)} \frac{\cos(r_{10} N)}{r_{10}^2}\, d\sigma + \sum_{i=1}^{k} \int_{(S_i)} \frac{\cos(r_{10} N)}{r_{10}^2}\, d\sigma. \tag{61}$$

If M lies in the subregion of (D_e) containing the infinitely distant point, then all the integrals on the right-hand side of the equation are zero. If M lies on (S_0) or in the interior of (D_i), then the first integral is equal to 2π or 4π, while the remaining integrals vanish. Finally, suppose that M lies on (S_i) or in the interior of the region bounded by (S_i). The first integral in (61) is then equal to 4π, the integrals over $(S_1), \ldots, (S_{i-1}), (S_{i+1}), \ldots, (S_k)$ are zero, and the integral over (S_i) is equal to -2π or -4π, since the normal of (S_i) is directed into the region (D_e), i.e., into the interior of the region bounded by (S_i).

§10. Another Proof of GAUSS' Formulas

In this section we shall give another proof of the formulas (60) and, in so doing, generalize them somewhat.

With this in mind, we consider the integral

$$\int\limits_{(\Sigma)} \frac{\cos(r_{10}N)}{r_{10}^2}\, d\sigma\,, \tag{62}$$

which is taken over a subregion (Σ) of a certain surface; we wish to determine the value of this integral at a point M.

We assume that the half lines radiating from M cut the subregion (Σ) in at most one point and that each half line forms an acute angle with the normal N of (Σ) at the point of intersection. We choose the point M itself as the origin of a system of polar coordinates and surround it with a unit sphere. Since $(r_{10}N)$ is the angle between the radius vector and the normal to (Σ) at the corresponding point of the surface, the quantity $\cos(r_{10}N)d\sigma$ is equal to the projection of the surface element $d\sigma$ of (Σ) onto the sphere about M with radius r_{10}. Then

$$d\sigma_0 = \frac{\cos(r_{10}N)}{r_{10}^2}\, d\sigma$$

is the corresponding surface element of the unit sphere. It follows therefore that the integral (62) is equal to the integral

$$\int\limits_{(\Sigma_0)} d\sigma_0\,, \tag{63}$$

where (Σ_0) is that portion of the unit sphere which is cut out by a cone with apex at M whose surface is generated by lines joining M to the boundary points of (Σ). The integral (63) is equal to the solid angle subtended by the surface (Σ) at the point M (Fig. 15).

If the radius vectors form obtuse angles with the normals of (Σ), then the integrals corresponding to (62) and (63) differ only in sign.

The side of a surface on which an observer stands upright along the direction of the normal we shall agree to call negative; the opposite side we shall call positive.

The results obtained may then be formulated as follows: the integral (62) is equal to the angle subtended by (Σ) at the point M with positive or

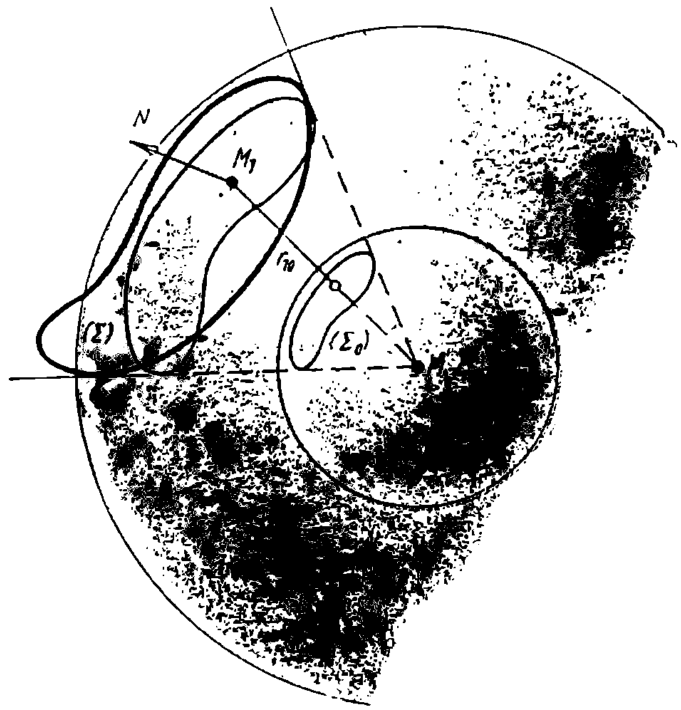

Fig. 15

negative sign depending on whether M is on the positive or negative side of the surface.

We now suppose that (Σ) is cut in several points by a half line radiating from M; we assume moreover that (Σ) can be decomposed into a finite number of subregions each of which satisfies the conditions of the previous case.[8] In this case the integral (62) is equal to the sum of the corresponding solid angles subtended by the different subregions of (Σ) at M, each solid angle taken with the appropriate sign.

Let (S) be a closed surface. If we move along a half line radiating from M at each point of intersection of the line with (S) we pass from the region (D_i) into the region (D_e) or vice versa. Points of entry into (D_i) and points of exit from (D_i) therefore alternate with one another; the last point is a point of exit (Fig. 16). At entry points the solid angle is negative, and at exit points it is positive. If M lies in (D_e) then the number of entry and exit points is the same, and the sum of the solid angles vanishes. Hence, the third of formulas (60) holds. If M lies in (D_i), then the first point is an exit point, and an equal number of exit and entry points follow. This holds for all directions, whence the validity of the first of formulas (60) follows easily.

[8] This condition is not satisfied by certain LYAPUNOV surfaces.

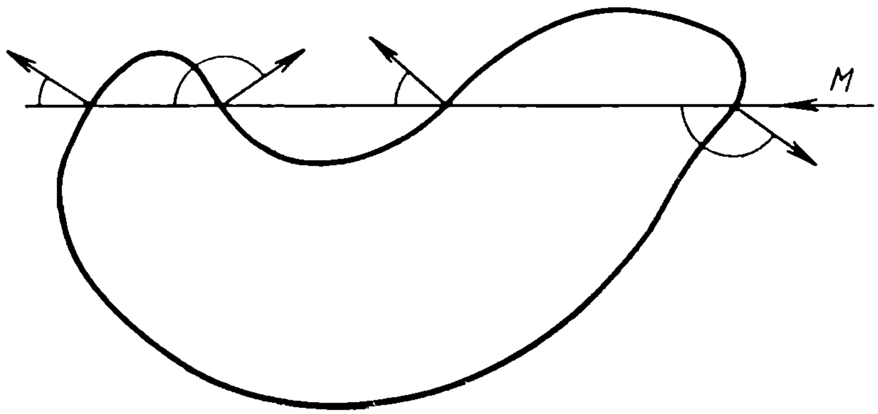

Fig. 16

The proof of the second of the formulas (60) is somewhat more difficult, and we shall omit this proof.

Appendix to Chapter I

Introducing curvilinear coordinates on (S) u = const., v = const., the first of equations (26) yields the equation

$$D_x\mu = \frac{1}{E}\frac{\partial(f)}{\partial u}\frac{\partial x}{\partial u} + \frac{1}{G}\frac{\partial(f)}{\partial v}\frac{\partial x}{\partial v}$$

with

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2,$$

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2.$$

The well-known formulas of O. Rodrigues imply that

$$\frac{\partial \cos(Nx)}{\partial u} = -\frac{1}{R_1}\frac{\partial x}{\partial u}, \qquad \frac{\partial \cos(Ny)}{\partial u} = -\frac{1}{R_1}\frac{\partial y}{\partial u},$$

and

$$\frac{\partial \cos(Nx)}{\partial v} = -\frac{1}{R_2}\frac{\partial x}{\partial v}, \qquad \frac{\partial \cos(Ny)}{\partial v} = -\frac{1}{R_2}\frac{\partial y}{\partial v}.$$

If one puts successively $\mu = (f) = \cos(Nx)$ and

$$\mu = (f) = \cos(Ny),$$

then it follows that

$$D_x \cos(Nx) = -\frac{1}{ER_1}\left(\frac{\partial x}{\partial u}\right)^2 + \frac{1}{GR_2}\left(\frac{\partial x}{\partial v}\right)^2$$

and

$$D_x \cos(Ny) = -\frac{1}{ER_1}\frac{\partial y}{\partial u}\frac{\partial x}{\partial u} + \frac{1}{GR_2}\frac{\partial y}{\partial v}\frac{\partial x}{\partial v}.$$

If one now makes use of the fact that

$$\cos(L_1x) = \frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}, \qquad \cos(L_1y) = \frac{1}{\sqrt{E}}\frac{\partial y}{\partial u},$$

$$\cos(L_2x) = \frac{1}{\sqrt{G}}\frac{\partial x}{\partial v}, \qquad \cos(L_2y) = \frac{1}{\sqrt{G}}\frac{\partial y}{\partial v},$$

one obtains formulas (27).

CHAPTER II

POTENTIAL THEORY

§1. The Potential of a Simple Layer

We suppose that a region (D_i) is bounded by a finite number of LYAPUNOV surfaces. We denote the boundary of the region (D_i) by (S) and the coordinates of the point M_2 of the surface (S) by ξ, η, and ζ.

Let M be a point with coordinates (x,y,z) and let r_{20} be the distance from the point M to the point M_2, where we assume that the segment MM_2 is oriented from M to M_2. Finally, let μ be an integrable function on (S). The function

$$V = \int_{(S)} \mu \frac{d\sigma}{r_{20}} \tag{1}$$

is called the *potential of a simple layer*; μ is called the density of the layer.

If M does not lie on (S), then V obviously possesses derivatives with respect to x, y, and z:

$$\frac{\partial V}{\partial x} = \int_{(S)} \mu \frac{\xi - x}{r_{20}^3} d\sigma,$$

$$\cdots\cdots\cdots,$$

$$\frac{\partial V}{\partial z} = \int_{(S)} \mu \frac{\zeta - z}{r_{20}^3} d\sigma$$

$$\frac{\partial^2 V}{\partial x^2} = \int_{(S)} \mu \frac{\partial^2 \frac{1}{r_{20}}}{\partial x^2} d\sigma,$$

$$\cdots\cdots\cdots\cdots.$$

Since the LAPLACE operator applied to the function $\dfrac{1}{r_{20}}$ gives zero, we have:

$$\Delta V = \int_{(S)} \mu \cdot \Delta \frac{1}{r_{20}} d\sigma = 0,$$

i.e., V is a harmonic function in any finite region which together with its boundary is contained in the interior of (D_i) or in the interior of (D_e).

If R denotes the distance of the point M from a fixed point of space, then from the formulas for the potential V and its first derivatives one easily sees that as the point M goes to infinity V goes to zero as the first power of $\frac{1}{R}$, while $\left|\frac{\partial V}{\partial x}\right|$, $\left|\frac{\partial V}{\partial y}\right|$, and $\left|\frac{\partial V}{\partial z}\right|$ go to zero as the second power of $\frac{1}{R}$. From this it follows that V is also harmonic in any infinite region which together with its boundary is contained in (D_e).

We now wish to show that the integral (1) converges for M on (S) under the condition that μ is bounded and integrable.

Let M be a point of (S). Let (σ_1) and (σ_2) denote subregions of (S) inside a LYAPUNOV sphere about M cut out by circular cylinders of radii ϱ_1 and ϱ_2 $(\varrho_2 < \varrho_1)$ with the normal at M as axis (Fig. 17). To demonstrate the convergence of the integral (1) it is sufficient to show that the integral

$$\int\limits_{(\sigma_1-\sigma_2)} |\mu| \frac{d\sigma}{r_{20}}$$

for $\varrho_1 \to 0$ becomes arbitrarily small.

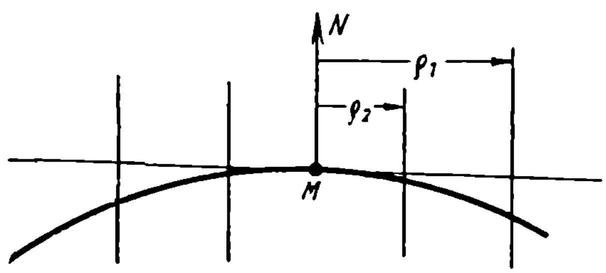

Fig. 17

Let A be an upper bound of $|\mu|$ on (S). Noting that the distance r_{20} is greater than its projection ϱ on the tangent plane at the point M and making use of estimate (16) of I, §1, we find that

$$\int\limits_{(\sigma_1-\sigma_2)} |\mu| \frac{d\sigma}{r_{20}} < A \cdot 2\int\limits_0^{2\pi}\int\limits_{\varrho_2}^{\varrho_1} \frac{\varrho\, d\varrho\, d\varphi}{\varrho} = 4\pi A(\varrho_1 - \varrho_2) < 4\pi A \varrho_1 .$$

This proves the assertion.

From the inequality obtained it follows that

$$\int\limits_{(\sigma_1)} \frac{|\mu|\, d\sigma}{r_{20}} < 4\pi A \varrho_1 . \tag{2}$$

We end this section with several remarks on inequalities which will be derived in subsequent sections.

Let μ be a function defined on (S) or in (D). It will mainly be assumed in what follows that μ is either bounded and integrable or bounded and H-continuous. In the first case A denotes an arbitrary upper bound for $|\mu|$. In the second case A denotes any of the upper bounds for $|\mu|$ which are greater than the coefficient in the condition for the H-continuity of μ; we then clearly have:

$$|\mu| < A, \qquad |\mu_1 - \mu_0| < A r_{10}^{\lambda},$$

where μ_1 and μ_0 are the values of μ at the points M_1 and M_0.

If the function μ is H-continuous with exponent λ, then it is obviously also H-continuous with an arbitrary exponent λ' such that $0 < \lambda' < \lambda$. Therefore, in the following λ will denote both the exponent of H-continuity of μ and the exponent in the second LYAPUNOV condition for LYAPUNOV surfaces.

In estimating various quantities we shall subsequently find that they are no larger than a sum involving the following functions of the variable δ:

$$c_1 A\,\delta, \quad c_2 A\,\delta^{\lambda}, \quad c_3 A\,\delta^{\lambda} \ln \frac{d}{\delta} \quad \left(0 < \delta \leq \frac{d}{2}, \quad 0 < \lambda \leq 1\right);$$

here c_1, c_2, and c_3 are constants which depend on the shape of the surface or region.

If only a sum of the first two functions enters the estimate, then since

$$c_1 A\,\delta = c_1 A\,\delta^{1-\lambda}\delta^{\lambda} \leq c_1' A\,\delta^{\lambda}, \qquad c_1' = c_1\left(\frac{d}{2}\right)^{1-\lambda},$$

the quantity to be estimated is no larger than $(c_1' + c_2)\,A = aA\delta^{\lambda}$ with $0 < \lambda \leqq 1$. If the third function also enters the estimate, then the quantity to be estimated is no larger than a number of the form $aA\delta^{\lambda'}$ where λ' is an arbitrary number in the interval $0 < \lambda' < \lambda$, and the magnitude of the coefficient a depends on the choice of λ'. Since for $\nu > 0$,

$$\delta^{\nu} \ln \frac{d}{\delta} = d^{\nu}\left(\frac{\delta}{d}\right)^{\nu} \ln \frac{d}{\delta} \leq d^{\nu} \cdot \operatorname*{Max}_{0<x\leq 1}\left(x^{\nu} \ln \frac{1}{x}\right) = \frac{d^{\nu}}{e\nu},$$

we have, putting $\nu = \lambda - \lambda'$:

$$c_3 A\,\delta^{\lambda} \ln \frac{d}{\delta} = c_2 A\,\delta^{\lambda'} \cdot \delta^{\nu} \ln \frac{d}{\delta} \leq \left(\frac{c_3 d^{\nu}}{e\nu}\right) A\,\delta^{\lambda'} = c_3' A\,\delta^{\lambda'}$$

with

$$c_3' = \frac{c_3 d^{\nu}}{e\nu}.$$

Our assertion now follows with $a = c_1\left(\frac{d}{2}\right)^{1-\lambda'} + c_2\left(\frac{d}{2}\right)^{\lambda-\lambda'} + c_3'$

Subsequently when in estimating some quantity a summand of the third type appears, we shall write the result of the estimate in the form $aA\delta^{\lambda'}$

without each time mentioning the arbitrary choice of λ' and the dependence of the coefficient a on the choice of λ'.

If no special restriction is made we shall subsequently assume that $\lambda < 1$, since the case $\lambda = 1$ requires unnecessary restrictions without, for the most part, leading to stronger results.

§2. Continuity of the Potential of the Simple Layer

Theorem. *If the density μ of the potential of a simple layer is a bounded and integrable function, then the potential is H-continuous in the entire space.*

Proof. From the observations of §1 it follows that V is H-continuous in any region which is entirely contained in (D_i) or (D_e). It remains therefore to consider points which lie in a neighborhood of the boundary.

We suppose that the point M_0 lies on (S) and that the point M_1, whose distance from M_0 we denote by δ, lies either on (S) or on the normal to (S) at the point M_0. For brevity, we consider both cases together; we denote the first case by (a) and the second by (b). Let r_{20} and r_{21} be the distances of the points M_0 and M_1 from the integration variable M_2. We surround the point M_0 with a LYAPUNOV sphere and denote by (Σ) the subregion of (S) lying inside the sphere. We assume first that δ is less than $\frac{d}{2}$.

If we denote by V_{M_0} and V_{M_1} the values of V at the points M_0 and M_1, then we can write:

$$V_{M_0} - V_{M_1} = \int\limits_{(S-\Sigma)} \mu \frac{d\sigma}{r_{20}} - \int\limits_{(S-\Sigma)} \mu \frac{d\sigma}{r_{21}} + \int\limits_{(\Sigma)} \mu \frac{d\sigma}{r_{20}} - \int\limits_{(\Sigma)} \mu \frac{d\sigma}{r_{21}} \cdot \tag{3}$$

The integral

$$\int\limits_{(S-\Sigma)} \mu \frac{d\sigma}{r_{21}}$$

is a continuous function of M_1 possessing derivatives of arbitrary order as long as the distance from M_1 to M_0 is less than $\frac{d}{2}$; in this case r_{21} remains greater than $\frac{d}{2}$. Hence,

$$\left| \int\limits_{(S-\Sigma)} \mu \frac{d\sigma}{r_{20}} - \int\limits_{(S-\Sigma)} \mu \frac{d\sigma}{r_{21}} \right| < a A \delta;$$

a is here a positive constant which depends only on (S), while A denotes an upper bound for $|\mu|$.

To prove the theorem it now suffices to study the difference

$$I = \int\limits_{(\Sigma)} \mu \frac{d\sigma}{r_{20}} - \int\limits_{(\Sigma)} \mu \frac{d\sigma}{r_{21}}\,. \tag{4}$$

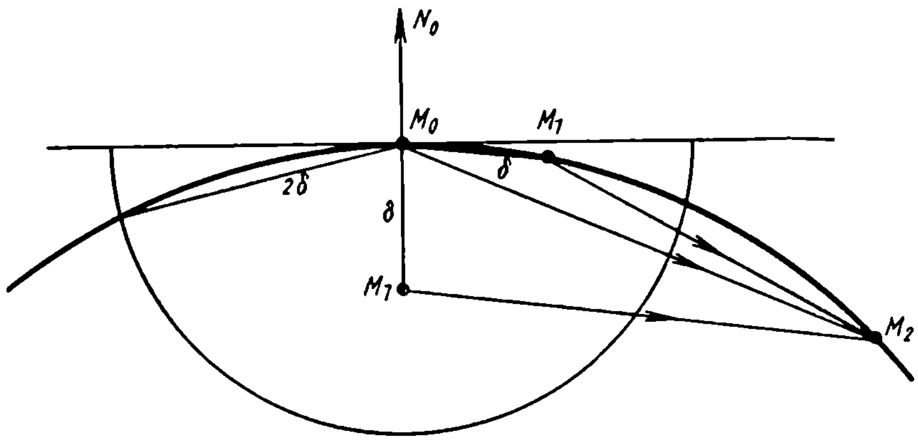

Fig. 18

We draw a sphere of radius 2δ about the point M_0 and denote the subregion of (S) lying inside the sphere by (σ) (Fig. 18). We can then write:

$$I = \int\limits_{(\Sigma-\sigma)} \mu \left(\frac{1}{r_{20}} - \frac{1}{r_{21}}\right) d\sigma + \int\limits_{(\sigma)} \mu \frac{d\sigma}{r_{20}} - \int\limits_{(\sigma)} \mu \frac{d\sigma}{r_{22}}\,. \tag{5}$$

From this it follows that

$$|I| \leq \left| \int\limits_{(\Sigma-\sigma)} \mu \left(\frac{1}{r_{20}} - \frac{1}{r_{21}}\right) d\sigma \right| + \left| \int\limits_{(\sigma)} \mu \frac{d\sigma}{r_{20}} \right| + \left| \int\limits_{(\sigma)} \mu \frac{d\sigma}{r_{21}} \right|. \tag{6}$$

Inequality (2) gives immediately:

$$\left| \int\limits_{(\sigma)} \mu \frac{d\sigma}{r_{20}} \right| < 4\pi A (2\delta), \tag{7}$$

since (σ) is contained in that subregion of the surface cut out from (Σ) by the circular cylinder with radius 2δ having the normal at M_0 as axis.

In case (a) one can estimate the last integral in (6) with the help of inequality (2). For this purpose we consider the sphere of radius 3δ with M_1 as origin; let (σ_1) be the subregion of (S) contained in the interior of this sphere. Since (σ) is contained in (σ_1), we have:

$$\left| \int\limits_{(\sigma)} \mu \frac{d\sigma}{r_{21}} \right| \leq \int\limits_{(\sigma)} |\mu| \frac{d\sigma}{r_{21}} \leq \int\limits_{(\sigma_1)} |\mu| \frac{d\sigma}{r_{21}} < 4\pi A (3\delta)\,. \tag{8}$$

In case (b) we estimate this integral directly. We find:

$$\left| \int\limits_{(\sigma)} \mu \frac{d\sigma}{r_{21}} \right| \leq \int\limits_{(\sigma)} |\mu| \frac{d\sigma}{r_{21}} < A \;\; 2\int_0^{2\pi} \int_0^{2\delta} \frac{\varrho\, d\varrho\, d\varphi}{\varrho} = 4\pi A (2\delta)\,, \tag{8'}$$

for in this case ϱ, as the projection of r_{20} on the tangent plane of (S) at the point M_0, is no greater than r_{21}.

We have now found that the sum of the absolute values of the last two integrals in (6) is in each case no greater than a number of the form $aA\delta$. We now turn to the first integral in (6). If we consider the triangle formed by r_{20} and r_{21} with third side δ, then the inequality $-\delta \leqq r_{21} - r_{20} \leqq \delta$ holds, whence the inequality

$$1 - \frac{\delta}{r_{20}} \leq \frac{r_{21}}{r_{20}} \leq 1 + \frac{\delta}{r_{20}}$$

follows.

If M_2 lies on $(\Sigma - \sigma)$, then $r_{20} > 2\delta$; hence

$$\frac{1}{2} < \frac{r_{21}}{r_{20}} < \frac{3}{2}, \quad \text{if } M_2 \text{ lies on } (\Sigma - \sigma). \tag{9}$$

We shall subsequently make frequent use of inequality (9).

With the help of this inequality we find:

$$\left|\frac{1}{r_{20}} - \frac{1}{r_{21}}\right| = \frac{|r_{21} - r_{20}|}{r_{21}\, r_{20}} < \frac{\delta}{r_{21} r_{20}} < \frac{2\delta}{r_{20}^2}.$$

If one joins the point M_0 with a point of the boundary curve of (σ), the projection of this segment on the tangent plane at M_0 is equal to $2\delta\cos\alpha$ where α is the angle between the segment and the tangent plane. Since inside the LYAPUNOV sphere the surface (S) is cut twice by this segment, it follows that $\alpha < \frac{\pi}{2} - \omega$ and hence $\cos\alpha > \sin\omega > \frac{1}{2}$. This projection is hence greater than δ. From this it follows that $(\Sigma - \sigma)$ is projected into a region of the tangent plane lying between circles of radii δ and d with center at M_0.

Introducing cylindrical coordinates, which we have already used twice previously, we find that r_{20} is no less than ϱ:

$$\left|\int\limits_{(\Sigma-\sigma)} \mu\left(\frac{1}{r_{20}} - \frac{1}{r_{21}}\right) d\sigma\right| \leq \int\limits_{(\Sigma-\sigma)} |\mu| \left|\frac{1}{r_{20}} - \frac{1}{r_{21}}\right| d\sigma$$

$$< 2\delta \cdot A \cdot 2\int\limits_0^{2\pi}\int\limits_\delta^d \frac{\varrho\, d\varrho\, d\varphi}{\varrho^2} = 8\pi A\delta \ln\frac{d}{\delta} \leq aA\delta^{\lambda'}.$$

We therefore arrive at the inequality

$$|V_{M_0} - V_{M_1}| < a_1 A\delta^{\lambda'} \qquad (\lambda' < 1).$$

We now suppose that M_1 does not lie on the normal to (S) at the point M_0. If the distance $|M_0M_1|$ is equal to δ, then the distance δ_1 of the point M_1 from (S) is less than δ. Let M' be the point of (S) nearest the

point M_1 (Fig. 19). The distance δ_2 between the points M_0 and M' is then less than $\delta_1 + \delta < 2\delta$.

Since

$$V_{M_0} - V_{M_1} = V_{M_0} - V_{M'} + V_{M'} - V_{M_1},$$

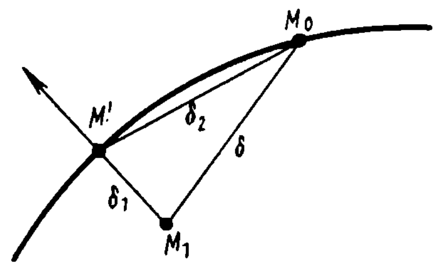

Fig. 19

we have:

$$|V_{M_0} - V_{M_1}| \leq |V_{M_0} - V_{M'}| + |V_{M'} - V_{M_1}| < a_1 A(\delta_2^{\lambda'} + \delta_1^{\lambda'}) < a_2 A \delta^{\lambda'}.$$

Case (a) shows that the potential V is H-continuous on (S). We wish to prove that V is H-continuous in the entire space. If one of the points M_0 or M_1 lies on (S) it has been shown that the difference $|V_{M_0} - V_{M_1}|$ can be estimated by the quantity $aA\delta^{\lambda'}$. To prove the H-continuity of V it remains only to consider the case in which neither M_0 nor M_1 lies on (S).

We assume that the point M_0 is the nearer of the two to the surface (S). Let M_0' be the point of (S) nearest the point M_0. M_0 then lies on the normal to (S) at the point M_0'. Let δ_1 denote the distance between M_0 and M_0' and δ the distance between M_0 and M_1. If $\delta_1 \geqq \frac{d}{2}$ then the difference

$$V_{M_0} - V_{M_1} = \int\limits_{(S)} \mu \frac{d\sigma}{r_{20}} - \int\limits_{(S)} \mu \frac{d\sigma}{r_{21}} \tag{3'}$$

can obviously be estimated in absolute value by $aA\delta$, since the integrands possess bounded derivatives in a neighborhood of the points M_0 and M_1.

We now suppose that $\delta_1 < \frac{d}{2}$. If $\delta_1 \leqq 2\delta$ then the distance δ_2 between M_1 and M_0' is no greater than 3δ and hence

$$|V_{M_0} - V_{M_1}| \leq |V_{M_0} - V_{M_0'}| + |V_{M_0'} - V_{M_1}| < aA\delta_1^{\lambda'} + aA\delta_2^{\lambda'} < bA\delta^{\lambda'}.$$

It remains to consider the case in which $\delta_1 > 2\delta$. In this case all points of (S) lie outside a sphere of radius 2δ with origin M_0. If r' is the distance of a certain point of the segment M_0M_1 from a point of the surface M_2, then using inequality (9), which is applicable here, we have: $r' > \frac{1}{2} r_{20}$.

From the relation

$$\frac{1}{r_{21}} - \frac{1}{r_{20}} = (x_1 - x_0)\left(\frac{\partial \frac{1}{r}}{\partial x}\right)_{M'} + (y_1 - y_0)\left(\frac{\partial \frac{1}{r}}{\partial y}\right)_{M'} + (z_1 - z_0)\left(\frac{\partial \frac{1}{r}}{\partial z}\right)_{M'},$$

in which (x_0,y_0,z_0) and (x_1,y_1,z_1) denote the coordinates of the points M_0 and M_1 respectively and the derivatives are taken at a point M' of the segment M_0M_1, one can conclude that

$$\left|\frac{1}{r_{21}} - \frac{1}{r_{20}}\right| < \frac{\sqrt{3}\,\delta}{r'^2} < \frac{4\sqrt{3}\,\delta}{r_{20}^2}$$

Hence

$$|V_{M_0} - V_{M_1}| < 4\sqrt{3}\,A\,\delta\int\limits_{(S)}\frac{d\sigma}{r_{20}^2}.$$

Now let (Σ) be the subregion of (S) inside a LYAPUNOV sphere about M_0' and (σ) be the subregion of (S) inside a sphere of radius 2δ with origin at M_0'.

On (S) the inequality $r_{20} \geqq \delta_1 > 2\delta$ holds, and hence

$$\delta\int\limits_{(\sigma)}\frac{d\sigma}{r_{20}^2} < \frac{\delta}{4\delta^2}\int\limits_{(\sigma)} d\sigma < \frac{1}{4\delta}\,2\cdot 2\pi\int\limits_0^{2\delta}\varrho\,d\varrho = 2\pi\delta,$$

$$\delta\int\limits_{(\Sigma-\sigma)}\frac{d\sigma}{r_{20}^2} < 4\pi\delta\int\limits_{\delta}^{d}\frac{\varrho\,d\varrho}{\varrho^2} = 4\pi\delta\ln\frac{d}{\delta} < a\,\delta^{\lambda'} \qquad (\lambda' < 1),$$

$$\delta\int\limits_{(S-\Sigma)}\frac{d\sigma}{r_{20}^2} < \delta\frac{4}{d^2}\int\limits_{(S-\Sigma)} d\sigma < \frac{4S}{d^2}\,\delta,$$

since off (Σ) $r_{20} > \frac{d}{2}$.

From this it follows that our assertion has been proved for all possible locations of the points M_0 and M_1. Hence, for any location of the points M_0 and M_1 we have: $|V_{M_1} - V_{M_0}| < aA\delta^{\lambda'}$.

§3. Three Theorems on the Potential of the Double Layer

Let $M_0(x,y,z)$ be a certain point of space. The integral

$$W = \int\limits_{(S)}\mu\,\frac{\cos(r_{20}\,N)}{r_{20}^2}\,d\sigma, \tag{10}$$

wherein $M_2(\xi,\eta,\zeta)$ is the integration variable, $\mu(\xi,\eta,\zeta)$ is a function defined on (S), and N is the normal to (S) at the point M_2, is called the *potential of a double layer* with density μ.

If μ is assumed to be integrable, then the integral (10) clearly exists for M_0 in the interior of (D_i) or (D_e). If M_0 is an arbitrary point of (S), then —as shown in I, §6—the boundedness of μ is a sufficient condition for the convergence of the integral (10).

We now wish to show that under the condition that μ be integrable W possesses continuous derivatives of arbitrary order at any interior point of (D_i) or (D_e) and satisfies the LAPLACE equation

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = 0$$

Integral (10) may be written in the following form:

$$W = \int\limits_{(S)} \mu \cos(N\xi) \frac{\xi - x}{r_{20}^3} d\sigma + \int\limits_{(S)} \mu \cos(N\eta) \frac{\eta - y}{r_{20}^3} d\sigma + \int\limits_{(S)} \mu \cos(N\zeta) \frac{\zeta - z}{r_{20}^3} d\sigma .$$

If one compares this expression with the formulas for the first derivatives of the potential of a simple layer given in §1, one sees that the last integrals are the derivatives with respect to x, y, and z of potentials of simple layers with densities $\mu \cos(N\xi)$, $\mu \cos(N\eta)$, and $\mu \cos(N\zeta)$ respectively.

Since the potential of a simple layer possesses derivatives of any order at any interior point of (D_i) or (D_e) and satisfies the LAPLACE equation, this clearly also holds for any derivative and hence also for W.

Moreover, as the point M_0 moves to infinity $|W|$ goes to zero as the second power of $\frac{1}{R}$ and each of the quantities $\left|\frac{\partial W}{\partial x}\right|$, $\left|\frac{\partial W}{\partial y}\right|$, and $\left|\frac{\partial W}{\partial z}\right|$ as the third power of $\frac{1}{R}$, where R denotes the distance from the point M_0 to a certain fixed point of space. W is therefore a harmonic function in any (finite or infinite) region which together with its boundary is contained in (D_i) or (D_e).

We agree to denote the value of W at a point of the surface by $\overline{W}$ and now prove the following two theorems.

Theorem 1. *If μ is bounded and integrable, then $\overline{W}$ is an H-continuous function of the point M_0 of (S).*

Theorem 2. *If μ is continuous on (S) and if the point M_1 lies in the interior of (D_i) or in the interior of (D_e), then as M_1 approaches the point M_0 of (S)*

$$\lim W_1 = W_i = \overline{W_0} + 2\pi\mu_0 \quad \text{or} \quad \lim W_1 = W_e = \overline{W_0} - 2\pi\mu_0 \qquad (11)$$

respectively.

We here denote by W_1 the value of W at the point M_1 and by μ_0 and $\overline{W}_0$ the values of μ and $\overline{W}$ at the point M_0. Henceforth W_i and W_e shall

denote the boundary values of W_1 as the point M_1 approaches the boundary of (D_i) from the inner and outer sides respectively.

Formula (11) demonstrates the validity of the equations

$$W_i - W_e = 4\pi\mu_0, \quad W_i + W_e = 2\overline{W}_0. \tag{12}$$

Corollary to Theorem 2. *If μ is H-continuous on (S), i.e., if*

$$|\mu - \mu_0| < A r_{20}^{\lambda}, \tag{13}$$

where r_{20} denotes the distance from M_2 to M_0, then the following inequalities

$$|W_1 - \overline{W}_0 - 2\pi\mu_0| < aA\delta^{\lambda},$$
$$|W_1 - \overline{W}_0 + 2\pi\mu_0| < aA\delta^{\lambda};$$

here δ is the distance of the point M_1 lying in either (D_i) or (D_e) from the point M_0.

Proof of the Theorems. Just as in §2 we treat both theorems together and denote the case in which M_1 lies on (S) by (*a*) and the case in which M_1 does not lie on (S) by (*b*). In the following we make use of the same notation as in §2. First of all, in the case in which the point M_1 does not lie on (S) we assume that it lies on the normal to (S) at the point M_0. We assume moreover that the distance δ between the points M_0 and M_1 is less than $\frac{d}{2}$. We recall further that the function μ satisfies the following conditions: in case (*a*) the function μ is bounded and its absolute value is less than a certain number A, in case (*b*) μ is continuous, and in the case of the corollary μ is H-continuous.

We begin with a remark concerning the case (*b*). Let (σ_0) be the subregion of (S) cut out by a sphere about M_0 with radius R, and let $\Theta(R)$ be the maximum of $|\mu - \mu_0|$ on (σ_0). Hence, if M lies on (σ_0)

$$|\mu - \mu_0| \leq \Theta(R). \tag{14}$$

If μ is H-continuous on (S), then

$$\Theta(R) < aR^{\lambda}.$$

If μ is only continuous on (S), one may conclude only that $\lim\limits_{R\to 0} \Theta(R) = 0$. $\Theta(R)$ is an increasing function of R, for if $R' > R$ (σ_0) is contained in the subregion (σ_0') corresponding to R'.

We assume that M_1 lies on (S) or on the normal to (S) at the point M_0. We then have:

$$\begin{aligned} W_1 - \overline{W} &= \int_{(S)} \mu \frac{\cos(r_{21} N)}{r_{21}^2} d\sigma - \int_{(S)} \mu \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma \\ &= \int_{(S)} (\mu - \mu_0) \frac{\cos(r_{21} N)}{r_{21}^2} d\sigma - \int_{(S)} (\mu - \mu_0) \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma \\ &+ \mu_0 \int_{(S)} \frac{\cos(r_{21} N)}{r_{21}^2} d\sigma - \mu_0 \int_{(S)} \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma . \end{aligned} \tag{15}$$

From the properties of GAUSS' integral the last integral in (15) is equal to 2π, and hence the last term in (15) is equal to $-2\pi\mu_0$. In case (a) the next to the last term in (15) is equal to $2\pi\mu_0$ so that the sum of the last two terms vanishes. In case (b) the cases M_1 in (D_i) and M_1 in (D_e) must be distinguished. If M_1 lies in (D_i) the next to the last term is equal to $4\pi\mu_0$; if M_1 lies in (D_e) it is equal to zero. We now have that the sum of the last two terms in (15) is

$$\left.\begin{array}{ll} 2\pi\mu_0 - 2\pi\mu_0 = 0, & \text{for } M_1 \text{ on } (S), \\ 4\pi\mu_0 - 2\pi\mu_0 = 2\pi\mu_0, & \text{for } M_1 \text{ in } (D_i) \text{ and} \\ 0 - 2\pi\mu_0 = -2\pi\mu_0, & \text{for } M_1 \text{ in } (D_e) \end{array}\right\} \tag{16}$$

It still remains to estimate the following difference:

$$I = \int_{(S)} (\mu - \mu_0) \frac{\cos(r_{21} N)}{r_{21}^2} d\sigma - \int_{(S)} (\mu - \mu_0) \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma .$$

Just as in §2 we surround the point M_0 with a LYAPUNOV sphere and denote by (Σ) the subregion of (S) lying inside the sphere. We can then write

$$\begin{aligned} I &= \int_{(S-\Sigma)} (\mu - \mu_0) \frac{\cos(r_{21} N)}{r_{21}^2} d\sigma - \int_{(S-\Sigma)} (\mu - \mu_0) \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma \\ &+ \int_{(\Sigma)} (\mu - \mu_0) \frac{\cos(r_{21} N)}{r_{21}^2} d\sigma + \int_{(\Sigma)} (\mu - \mu_0) \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma . \end{aligned}$$

If M_2 lies on $(S - \Sigma)$, then r_{21} is greater than $\frac{d}{2}$, and the integral

$$\int_{(S-\Sigma)} (\mu - \mu_0) \frac{\cos(r_{21} N)}{r_{21}^2} d\sigma$$

together with its derivatives is continuous and bounded, since the distance from M_1 to M_0 is no greater than $\frac{d}{2}$. From this it follows that

$$\left| \int_{(S-\Sigma)} (\mu - \mu_0) \frac{\cos(r_{21} N)}{r_{21}^2} d\sigma - \int_{(S-\Sigma)} (\mu - \mu_0) \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma \right| < a A \delta ,$$

so that it only remains to estimate the difference

$$I_1 = \int_{(\Sigma)} (\mu - \mu_0) \frac{\cos(r_{21} N)}{r_{21}^2} d\sigma - \int_{(\Sigma)} (\mu - \mu_0) \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma .$$

Just as in §2 we draw a sphere of radius 2δ about M_0. It then follows that

$$\left.\begin{aligned} I_1 &= \int_{(\Sigma-\sigma)} (\mu - \mu_0) \left\{ \frac{\cos(r_{21} N)}{r_{21}^2} - \frac{\cos(r_{20} N)}{r_{20}^2} \right\} d\sigma \\ &\quad + \int_{(\sigma)} (\mu - \mu_0) \frac{\cos(r_{21} N)}{r_{21}^2} d\sigma - \int_{(\sigma)} (\mu - \mu_0) \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma ; \\ |I_1| &\le \left| \int_{(\Sigma-\sigma)} (\mu - \mu_0) \left\{ \frac{\cos(r_{21} N)}{r_{21}^2} - \frac{\cos(r_{20} N)}{r_{20}^2} \right\} d\sigma \right| \\ &\quad + \left| \int_{(\sigma)} (\mu - \mu_0) \frac{\cos(r_{21} N)}{r_{21}^2} d\sigma \right| + \left| \int_{(\sigma)} (\mu - \mu_0) \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma \right| . \end{aligned}\right\} \quad (17)$$

Since $|\mu - \mu_0| < 2A$, we find by making use of inequality (38) of I, §6 and the considerations of §2 that the last integral in (17) is smaller in absolute value than $cA(2\delta)^\lambda$ where c is a constant. In case (*a*) the next to the last integral in (17) is less than $cA(3\delta)^\lambda$, since (σ)—as noted in §2—is contained in the subregion (σ_2) of (S) lying inside the sphere of radius 3δ with origin M_1.

In case (*b*) we estimate this integral directly. On (σ)

$$|\mu - \mu_0| < \Theta(2\delta)$$

and $\cos(r_{21} N)$ is less than one in absolute value. If in the triangle $M_0 M_1 M_2$ we denote the angles at the corners M_0 and M_1 respectively by α and β, then we find that

$$\frac{r_{21}}{\delta} = \frac{\sin\alpha}{\sin\beta}, \quad r_{21} = \delta \frac{\sin\alpha}{\sin\beta} \ge \delta \sin\alpha > \delta \sin\omega > \frac{1}{2}\delta ,$$

for α is the angle between the normal to (S) at the point M_0 and a line passing through M_0 which cuts (S) twice inside the LYAPUNOV sphere.

From this follows the inequality

$$\begin{aligned} \left| \int_{(\sigma)} (\mu - \mu_0) \frac{\cos(r_{21} N)}{r_{21}^2} d\sigma \right| &< \frac{\Theta(2\delta)}{\delta^2} \, 2^2 \int_{(\sigma)} d\sigma \\ &< \frac{2^3 \Theta(2\delta)}{\delta^2} \int_0^{2\pi} \int_0^{2\delta} \varrho \, d\varrho \, d\varphi = 32\pi\Theta(2\delta) , \end{aligned} \quad (18)$$

from which one concludes that as $\delta \to 0$ the integral under consideration becomes arbitrarily small.

If μ is H-continuous, then the absolute value of the integral can be estimated by a quantity of the form $aA\delta^\lambda$. Hence, in case (a) and in the case of the corollary the last two integrals in (17) can be estimated in absolute value by the quantity $aA\delta^\lambda$; in case (b) this sum becomes arbitrarily small as $\delta \to 0$.

We now consider the first term of (17). We have:

$$\frac{\cos(r_{21}N)}{r_{21}^2} - \frac{\cos(r_{20}N)}{r_{20}^2} = \frac{r_{21}\cos(r_{21}N)}{r_{21}^3} - \frac{r_{20}\cos(r_{20}N)}{r_{20}^3}$$

$$= \frac{r_{21}\cos(r_{21}N) - r_{20}\cos(r_{20}N)}{r_{21}^3} + r_{20}\cos(r_{20}N)\left\{\frac{1}{r_{21}^3} - \frac{1}{r_{20}^3}\right\}$$

$$= \frac{\delta\cos(r_{01}N)}{r_{21}^3} + r_{20}\cos(r_{20}N)\left\{\frac{1}{r_{21}^3} - \frac{1}{r_{20}^3}\right\}. \tag{19}$$

Indeed, the vector from M_1 to M_2 is in all cases equal to the sum of the vectors from M_1 to M_2 and from M_0 to M_2; hence

$$r_{21}\cos(r_{21}N) = r_{20}\cos(r_{20}N) + \delta\cos(r_{01}N).$$

We discuss the cases (a) and (b) separately.

Denoting the normal to (S) at M_0 by N_0, we have in case (a):

$$\cos(r_{01}N) = \cos(r_{01}N_0) + [\cos(r_{01}N) - \cos(r_{01}N_0)];$$

but

$$|(r_{01}N) - (r_{01}N_0)| \leq (N_0N)$$

and hence

$$|\cos(r_{01}N) - \cos(r_{01}N_0)| \leq (N_0N) < Er_{20}^\lambda.$$

Moreover, from inequality (17) of I, §1

$$|\cos(r_{01}N_0)| < Er_{01}^\lambda = E\delta^\lambda$$

and hence

$$\delta\,|\cos(r_{01}N)| < E\delta^{1+\lambda} + E\delta r_{20}^\lambda.$$

In estimating the integral corresponding to the last term on the right-hand side of (19), on the basis on inequality (9) we may proceed as follows:

$$\left|\int_{(\Sigma-\sigma)}(\mu - \mu_0)\frac{\delta\cos(r_{01}N)}{r_{21}^3}\,d\sigma\right|$$

$$< 2AE\delta\cdot 2\cdot 2^3\int_0^{2\pi}\int_\delta^d \frac{\varrho\,d\varrho\,d\varphi}{\varrho^{3-\lambda}} + 2AE\,\delta^{1+\lambda}\cdot 2\cdot 2^3\int_0^{2\pi}\int_\delta^d\frac{\varrho\,d\varrho\,d\varphi}{\varrho^3}$$

$$= c_1A\delta\left\{\frac{1}{\delta^{1-\lambda}} - \frac{1}{d^{1-\lambda}}\right\} + c_2A\,\delta^{1+\lambda}\left\{\frac{1}{\delta} - \frac{1}{d}\right\} < cA\,\delta^\lambda.$$

We now consider the case (*b*). Since $\Theta(R)$ is a nondecreasing function, we may write, using inequality (12) of I, §1:

$$|\mu - \mu_0| \leq \Theta(r_{20}) \leq \Theta(2\varrho).$$

Introducing cylindrical coordinates, we find that

$$\left| \int_{(\Sigma-\sigma)} (\mu - \mu_0) \frac{\delta \cos(r_{01} N)}{r_{21}^3} d\sigma \right| < 2^3 \delta \cdot 2 \int_0^{2\pi} \int_\delta^d \frac{\Theta(2\varrho)\, \varrho\, d\varrho\, d\varphi}{\varrho^3}$$

$$= 2^3 4\pi\delta \int_\delta^d \frac{\Theta(2\varrho)}{\varrho^2} d\varrho.$$

We have here replaced r_{21} by $\frac{1}{2} r_{20}$ and r_{20} by ϱ.

As δ goes to zero, we find:

$$\lim_{\delta\to 0} \delta \int_\delta^d \frac{\Theta(2\varrho)}{\varrho^2} d\varrho = \lim_{\delta\to 0} \frac{\int_\delta^a \frac{\Theta(2\varrho)}{\varrho^2} d\varrho}{\frac{1}{\delta}} = \lim_{\delta\to 0} \frac{-\frac{\Theta(2\delta)}{\delta^2}}{-\frac{1}{\delta^2}} = \lim_{\delta\to 0} \Theta(2\delta) = 0.$$

In the case of the corollary the integral in question is less in absolute value than

$$2^3 \cdot 4\pi\delta A \int_\delta^d \frac{(2\varrho)^\lambda d\varrho}{\varrho^2} = \alpha A \delta \left\{ \frac{1}{\delta^{1-\lambda}} - \frac{1}{d^{1-\lambda}} \right\} < a A \delta^\lambda.$$

Hence, in case (*a*) and in the case of the corollary the integral in question can be estimated by the quantity $aA\delta^\lambda$; in case (*b*) the absolute value of the integral becomes arbitrarily small as $\delta \to 0$. We now turn to the last term on the right-hand side of (19); inequality (9) yields

$$\left| r_{20} \left\{ \frac{1}{r_{21}^3} - \frac{1}{r_{20}^3} \right\} \right| = \frac{|r_{20}^3 - r_{21}^3|}{r_{21}^3 r_{20}^2} = \frac{|r_{21} - r_{20}|\,(r_{20}^2 + r_{20} r_{21} + r_{21}^2)}{r_{21}^3 r_{20}^2}$$

$$< \frac{\delta\left(r_{20}^2 + \frac{3}{2} r_{20}^2 + \frac{9}{4} r_{20}^2\right)}{\left(\frac{1}{2}\right)^3 r_{20}^3 r_{20}^2} < \frac{a\delta}{r_{20}^3}.$$

On the basis of inequality (17) of I, §1,

$$|\cos(r_{20} N)| < E r_{20}^\lambda.$$

Hence, in each case

$$\left| \int\limits_{(\Sigma-\sigma)} (\mu - \mu_0)\, r_{20} \cos(r_{20} N) \left\{ \frac{1}{r_{21}^3} - \frac{1}{r_{20}^3} \right\} d\sigma \right|$$

$$< 2aAE\delta \int\limits_{(\Sigma-\sigma)} \frac{r_{20}^{\lambda} d\sigma}{r_{20}^3} < 2aAE\delta \cdot 2 \int\limits_0^{2\pi} \int\limits_{\delta}^{d} \frac{\varrho\, d\varrho\, d\varphi}{\varrho^{3-\lambda}} < cA\delta^{\lambda}.$$

In summary, we have established that the integral I_1 can in case (*a*) and in the case of the corollary be estimated by the quantity $aA\delta^{\lambda}$, while it becomes arbitrarily small in absolute value as $\delta \to 0$ in case (*b*). Theorem 1 and the corollary are herewith proved; Theorem 2 has now been proved for the case in which M_1 lies on the normal to (S) at the point M_0. By hypothesis, μ is continuous on (S). Since (S) is a closed and bounded set, μ is uniformly continuous on (S). It is therefore possible to find a function $\Theta(R)$ which is increasing ($\Theta(R) \to 0$ for $R \to 0$) such that inequality (14) is satisfied independent of the location of the point M_0 on (S). Collecting all the inequalities found for case (*b*) and, in order to have a specific case at hand, considering the case in which M_1 lies in (D_i), we find:

$$|W_1 - \overline{W}_0 - 2\pi\mu_0| < aA\delta + bA\delta^{\lambda} + c\Theta(2\delta) + q\delta \int\limits_{\delta}^{d} \frac{\Theta(2\varrho)}{\varrho^2} d\varrho = \psi(\delta);$$

$\psi(\delta)$ is here a continuous function which has limit zero as $\delta \to 0$. If the coefficient a is chosen sufficiently large, as we shall assume, $\psi(\delta)$ is an increasing function.

Now suppose that the point M_1 lies in (D_i) and that its distance from M_0 is δ. The distance δ_1 of the point M_1 from (S) is then no greater than δ. If M_2 is the point on (S) nearest M_1, then the distance between M_2 and M_0 is less than 2δ; taking account of the fact that M_1 lies on the normal to (S) at the point M_2, we obtain:

$$|W_1 - \overline{W}_0 - 2\pi\mu_0| \leq |W_1 - \overline{W}_2 - 2\pi\mu_2| + |\overline{W}_2 - \overline{W}_0| + 2\pi|\mu_2 - \mu_0|$$

$$< \psi(\delta_1) + aA\delta^{\lambda} + 2\pi\Theta(2\delta) \leq \psi(\delta) + aA\delta^{\lambda} + 2\pi\Theta(2\delta) = \varphi(\delta)$$

where $\varphi(\delta) \to 0$ for $\delta \to 0$. Theorem 2 is herewith proved.

From Theorem 2 it follows that *the potential of a double layer W with continuous density is uniformly continuous in (D_i) and (D_e) and is a harmonic function in the interior of (D_i) and in the interior of (D_e).*

To prove this, in the closed region composed of (D_i) and the boundary (S) we consider a function which is equal to W in (D_i) and is equal to W_i on (S). It is clear that W is continuous at every interior point of (D_i) even when the density μ is not continuous. Moreover, from Theorems 1 and 2 the function in question is continuous at every point of (S). This function is therefore continuous at every point of the finite closed region and is hence uniformly continuous. From this it follows that W is uniformly

continuous in (D_i). The behavior of the first derivatives of W as the point M_0 moves outward ensures that W is uniformly continuous outside every sphere containing the surface (S) in its interior. For the finite subregion of (D_e) inside the sphere the uniform continuity can be proved just as for (D_i). Hence, W is uniformly continuous in (D_e). Since W satisfies the LAPLACE equation and together with its derivatives goes to zero as the point M_0 goes to infinity, W is harmonic in the interior of (D_i) and in the interior of (D_e).

Addendum. We note that Theorem 2 can be proved for any arbitrary point M_0 lying in the interior of a subregion of (S) at which the function μ is continuous. Therefore, we have the

Theorem. *Suppose that the density μ of the potential of the double layer is integrable (though possibly unbounded). If in a certain subregion of (S) μ is continuous and if M_0 is an interior point of this subregion, then formulas* (11) *hold.*

We likewise have the theorem: *If μ is integrable on (S) and is bounded in a neighborhood of the point M_0, then $\overline{W}$ is continuous at M_0.*

Theorem 3. *If μ is H-continuous, then the potential of the double layer is H-continuous in (D_i) and in (D_e).*

Proof. To have a specific case in mind, we consider the case of the region (D_i) and denote by $W_i(M)$ the limit of W at the point M of (S).

By hypothesis,

$$|\mu| < A, \quad |\mu_1 - \mu_0| < A r_{10}^{\lambda},$$

and hence $\Theta(R) = AR^{\lambda}$. We therefore find that for simplicity we may restrict ourselves to the case $\lambda < 1$:

$$\psi(\delta) = aA\delta + bA\delta^{\lambda} + c\,\Theta(2\delta) + q\delta \int_{\delta}^{d} \frac{A(2\varrho)^{\lambda}}{\varrho^2}\, d\varrho < c_1 A \delta^{\lambda} \left(\delta \leq \frac{d}{2}\right),$$

and similarly $\varphi(\delta) < c_2 A\delta^{\lambda}$.

On the basis of the previous results we have therefore:

$$|W(M_1) - W_i(M)| < c_2 A \delta_1^{\lambda},$$

in the case that the distance between the point M_1 of the region (D_i) and the point M of the surface (S) is less than or equal to δ_1.

Let δ be the distance between the two points M_1 and M_2 belonging to (D_i). We wish to show that there exists a constant $c > 0$ such that

$$|W(M_1) - W(M_2)| < cA\delta^{\lambda}.$$

For this it is necessary to consider several possibilities for the positions

of M_1 and M_2. If $\delta \geqq \frac{d}{4}$, then since $|W| < aA$:

$$|W(M_1) - W(M_2)| < 2aA = A\frac{2a}{\left(\frac{d}{4}\right)^\lambda}\left(\frac{d}{4}\right)^\lambda = Ac\cdot\left(\frac{d}{4}\right)^\lambda \leq cA\delta^\lambda.$$

It is therefore sufficient to treat the case in which $\delta < \frac{d}{4}$.

We denote by δ_1 and δ_2 the distances of the points M_1 and M_2 from the surface (S), whereby we shall always assume that $\delta_1 \leqq \delta_2$. If now $\delta < \frac{d}{4}$ and $\delta_1 \geqq \frac{d}{2}$, then the points M_1 and M_2 and the line joining them lie in a region all points of which are at least a distance $\frac{d}{4}$ from (S). The absolute values of the first derivatives of W are clearly in this region bounded by a number of the form cA; hence, the absolute value of the difference $W(M_1) - W(M_2)$ is less than a number of the form $3cA\delta$.

It remains to treat the case in which $\delta_1 < \frac{d}{2}$ and $\delta < \frac{d}{4}$. Here, two additional cases are to be distinguished:

1. $\delta_1 < 2\delta$. If M is the point of (S) nearest M_1, then $|M_1M| = \delta_1$ and $|M_2M| \leqq |M_1M_2| + |M_1M| = \delta + \delta_1 < 3\delta$. Hence, it follows that

$$\begin{aligned}|W(M_1) - W(M_2)| &\leq |W(M_1) - W_i(M)| + |W_i(M) - W(M_2)| \\ &\leq c_3A(2\delta)^\lambda + c_3A(3\delta)^\lambda = c_4A\delta^\lambda.\end{aligned}$$

2. $\delta_1 \geqq 2\delta$. We denote by (Σ) the subregion of (S) lying inside the LYAPUNOV sphere about the point M_0 of (S) which is nearest M_1 and by (σ) the subregion of (Σ) contained in the interior of the sphere of radius 2δ about M_0. With M_3 as integration variable we then obtain:

$$\begin{aligned}W(M_2) - W(M_1) &= \int_{(S)} \mu\frac{\cos(r_{13}N_3)}{r_{13}^2}\,d\sigma_3 - \int_{(S)} \mu\frac{\cos(r_{23}N_3)}{r_{23}^2}\,d\sigma_3 \\ &= \int_{(S-\Sigma)} (\mu - \mu_0)\frac{\cos(r_{13}N_3)}{r_{13}^2}\,d\sigma_3 - \int_{(S-\Sigma)} (\mu - \mu_0)\frac{\cos(r_{23}N_3)}{r_{23}^2}\,d\sigma_3 \\ &+ \int_{(\Sigma-\sigma)} (\mu - \mu_0)\left\{\frac{\cos(r_{13}N_3)}{r_{13}^2} - \frac{\cos(r_{23}N_3)}{r_{23}^2}\right\}d\sigma_3 \\ &+ \int_{(\sigma)} (\mu - \mu_0)\frac{\cos(r_{13}N_3)}{r_{13}^2}\,d\sigma_3 - \int_{(\sigma)} (\mu - \mu_0)\frac{\cos(r_{23}N_3)}{r_{23}^2}\,d\sigma_3;\end{aligned}$$

where μ_0 denotes the value of μ at the point M_0.

Considering the fact that $r_{13} \geqq 2\delta$ and $r_{23} \geqq 2\delta$, we estimate the absolute values of the integrals over (σ) by numbers of the following form:

$$4\pi A\int_0^{2\delta}\varrho^\lambda\frac{\varrho\, d\varrho}{\delta^2}=\frac{4\pi A}{\delta^2}\cdot\frac{(2\delta)^{2+\lambda}}{2+\lambda}=A\frac{4\pi\, 2^{2+\lambda}}{2+\lambda}\delta^\lambda=aA\delta^\lambda.$$

The distances of the points M_1 and M_2 from $(S-\Sigma)$ are not less than $\frac{d}{4}$. Hence, the absolute value of the difference of the integrals over $(S-\Sigma)$ is no greater than a number of the form $cA\delta$. It remains only to estimate the integral over $(\Sigma-\sigma)$.

We have:

$$\left|\frac{\cos(r_{13}N_3)}{r_{13}^2}-\frac{\cos(r_{23}N_3)}{r_{23}^2}\right|\leq\left|\frac{r_{13}\cos(r_{13}N_3)-r_{23}\cos(r_{23}N_3)}{r_{13}^3}\right|$$
$$+\left|\cos(r_{23}N_3)\left(\frac{r_{23}}{r_{13}^3}-\frac{1}{r_{23}^2}\right)\right|\leq\frac{1}{r_{13}^3}\delta\,|\cos(r_{13}N_3)|$$
$$+\frac{|r_{23}-r_{13}|\,(r_{23}^2+r_{23}r_{13}+r_{13}^2)}{r_{13}^3r_{23}^2}\leq\frac{\delta}{r_{13}^3}+\delta\left(\frac{1}{r_{13}^3}+\frac{1}{r_{13}^2r_{23}}+\frac{1}{r_{13}r_{23}^2}\right).$$

Since the point M_3 on $(\Sigma-\sigma)$ lies outside the sphere of radius 2δ about M_1, it follows from inequality (9) that $r_{23}>\frac{1}{2}r_{13}$; moreover, obviously $r_{13}\geqq\varrho$, and hence the difference in question is in absolute value no greater than $\frac{8\delta}{\varrho^3}$. From this we obtain the following estimate for the absolute value of the integral over $(\Sigma-\sigma)$:

$$8\pi A\cdot 8\delta\int_\delta^d\varrho^\lambda\frac{\varrho\, d\varrho}{\varrho^3}=8\pi A\cdot 8\delta\frac{\delta^{\lambda-1}-d^{\lambda-1}}{1-\lambda}<\frac{64\pi}{1-\lambda}A\delta^\lambda.$$

Hence, in the case $\delta_1\geqq 2\delta$ the absolute value of the difference $W(M_1)-W(M_2)$ is no greater than a number of the form $cA\delta^\lambda$; the theorem is herewith proved. It should be noted that in the case $\lambda=1$ instead of estimates of the form $cA\delta^\lambda$ one obtains quantities of the form $c'A\delta^{\lambda'}$ where λ' is an arbitrary number in the interval $0<\lambda'<1$.

§4. On the Normal Derivative of the Potential of the Simple Layer

In §1 it was proved that the potential of the simple layer possesses continuous derivatives with respect to the coordinates x, y, and z of the point $M(x,y,z)$ lying in the interior of either (D_i) or (D_e). At a point M_0 of the surface (S) we construct the normal N_0 and compute the derivatives mentioned at a point M_1 of this normal (Fig. 20):

$$\frac{\partial V}{\partial x} = \int\limits_{(S)} \mu \frac{\xi - x_1}{r_{21}^3} d\sigma = \int\limits_{(S)} \mu \frac{\cos(r_{21} x)}{r_{21}^2} d\sigma, \ldots;$$

x_1, y_1, and z_1 here denote the coordinates of the point M_1.

We form the combination

$$\left(\frac{dV}{dn}\right)_{M_1} = \frac{\partial V}{\partial x}\cos(N_0 x) + \frac{\partial V}{\partial y}\cos(N_0 y) + \frac{\partial V}{\partial z}\cos(N_0 z)$$

$$= \int\limits_{(S)} \mu \frac{\cos(r_{21} x)\cos(N_0 x) + \cos(r_{21} y)\cos(N_0 y) + \cos(r_{21} z)\cos(N_0 z)}{r_{21}^2} d\sigma;$$

this leads to the formula

$$\left(\frac{dV}{dn}\right)_{M_1} = \int\limits_{(S)} \mu \frac{\cos(r_{21} N_0)}{r_{21}^2} d\sigma. \tag{20}$$

Assuming the continuity of μ, we now wish to show that (20) possesses a limit as M_1 approaches M_0 with M_1 remaining either in (D_i) or in (D_e). These two limits will in general differ; we denote them by

$$\frac{dV_i}{dn} \quad \text{and} \quad \frac{dV_e}{dn}.$$

It must be shown that the integral (20) converges when we replace the point M_1 by the point M_0 on (S). One hereby obtains a certain function of the point M_0 of the surface which we call the normal derivative of V and denote by

$$\frac{dV}{dn} = \int\limits_{(S)} \mu \frac{\cos(r_{20} N_0)}{r_{20}^2} d\sigma. \tag{21}$$

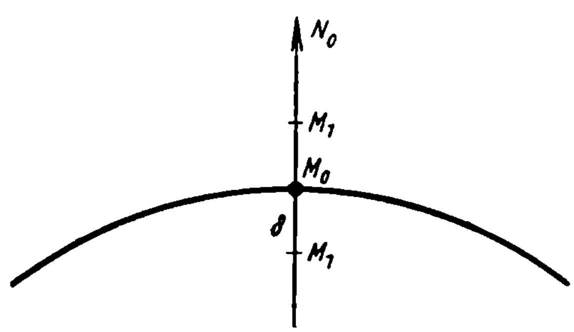

Fig. 20

Having introduced the function $\frac{dV}{dn}$, one can prove the equalities

$$\frac{dV_i}{dn} = \frac{dV}{dn} + 2\pi\mu_0, \qquad \frac{dV_e}{dn} = \frac{dV}{dn} - 2\pi\mu_0$$

where μ_0 denotes the value of μ at the point M_0.

These equations will be derived in §6. They yield the relations

$$\frac{dV_i}{dn} - \frac{dV_e}{dn} = 4\pi\mu_0, \qquad \frac{dV_i}{dn} + \frac{dV_e}{dn} = 2\frac{dV}{dn}.$$

We shall now prove that the integral (21) converges if the density μ is bounded and integrable. Just as in §1 (Fig. 17) we consider two circular cylinders with radii ϱ_1 and ϱ_2 ($\varrho_2 < \varrho_1$) and with axis N_0, and we denote by (σ_1) and (σ_2) the subregions of (S) cut out by these cylinders. To prove the convergence of the integral (21) it is sufficient to show that the integral

$$\int\limits_{(\sigma_1-\sigma_2)} |\mu| \frac{|\cos(r_{20}N_0)|}{r_{20}^2}\, d\sigma$$

becomes arbitrarily small as $\varrho_1 \to 0$.

From inequality (17) of I, §1 we have:

$$|\cos(r_{20}N_0)| < E r_{20}^{\lambda}.$$

Using cylindrical coordinates we therefore obtain:

$$\int\limits_{(\sigma_1-\sigma_2)} |\mu| \frac{|\cos(r_{20}N_0)|}{r_{20}^2}\, d\sigma < EA \int\limits_{(\sigma_1-\sigma_2)} \frac{r_{20}^{\lambda}}{r_{20}^2}\, d\sigma < EA\cdot 2\int\limits_0^{2\pi}\int\limits_{\varrho_2}^{\varrho_1} \frac{\varrho\, d\varrho\, d\varphi}{\varrho^{2-\lambda}}$$
$$= \frac{4\pi EA}{\lambda}(\varrho_1^{\lambda} - \varrho_2^{\lambda}) < aA\varrho_1^{\lambda}.$$

This completes the proof of the theorem.

From the last inequality it follows that

$$\left|\int\limits_{(\sigma_1)} \mu \frac{\cos(r_{20}N_0)}{r_{20}^2}\, d\sigma\right| < aA\varrho_1^{\lambda}, \tag{22}$$

where a is a constant depending only on the surface (S).

Remark. From the preceding considerations one may conclude that

$$\left|\int\limits_{(S)} \mu \frac{\cos(r_{20}N_0)}{r_{20}^2}\, d\sigma\right| < aA,$$

where A is an upper bound for $|\mu|$. If the sequence $\{\mu_n\}$ converges uniformly with increasing n to the limit function μ on (S), then it follows just as in I, §6 that

$$\lim_{n\to\infty} \int\limits_{(S)} \mu_n \frac{\cos(r_{20}N_0)}{r_{20}^2}\, d\sigma = \int\limits_{(S)} \mu \frac{\cos(r_{20}N_0)}{r_{20}^2}\, d\sigma.$$

§5. The Continuity of the Normal Derivative of the Potential of the Simple Layer

Theorem. *If the density* μ *of the potential of the simple layer is bounded and integrable, then the normal derivative*

$$\frac{dV}{dn} = \int_{(S)} \mu \frac{\cos(r_{20} N_0)}{r_{20}^2} \, d\sigma$$

is an H-continuous function of the point M_0 *of the surface* (S).

Proof. Let δ be the distance $|M_1 M_0|$. We wish to determine the difference of the values of the integral (21) at the points M_1 and M_0,

$$\int_{(S)} \mu \frac{\cos(r_{21} N_1)}{r_{21}^2} \, d\sigma - \int_{(S)} \mu \frac{\cos(r_{20} N_0)}{r_{20}^2} \, d\sigma, \tag{23}$$

where N_1 denotes the normal to (S) at the point M_1. For this purpose, we introduce the LYAPUNOV sphere about M_0 and write the difference (23) in the form

$$\begin{aligned}\int_{(S-\Sigma)} \mu \frac{\cos(r_{21} N_1)}{r_{21}^2} \, d\sigma &- \int_{(S-\Sigma)} \mu \frac{\cos(r_{20} N_0)}{r_{20}^2} \, d\sigma \\ &+ \int_{(\Sigma)} \mu \frac{\cos(r_{21} N_1)}{r_{21}^2} \, d\sigma - \int_{(\Sigma)} \mu \frac{\cos(r_{20} N_0)}{r_{20}^2} \, d\sigma .\end{aligned} \tag{24}$$

To estimate the first difference in (24) we consider the potential of the simple layer

$$V_1 = \int_{(S-\Sigma)} \mu \frac{1}{r_{21}} \, d\sigma .$$

If $r_{10} = \delta < \frac{d}{2}$, then $r_{21} > \frac{d}{2}$; the potential V then possesses continuous derivatives of arbitrary order at the point M_1, and the absolute values of the potential and its first and second derivatives can be estimated by a quantity of the form aA. The difference of the first two integrals in (24) can be written in the following manner:

$$\begin{aligned}\left(\frac{\partial V_1}{\partial n_1}\right)_{M_1} - \left(\frac{\partial V_1}{\partial n_0}\right)_{M_0} &= \left(\left(\frac{\partial V_1}{\partial x}\right)_{M_1} \cos(N_1 x) + \cdots\right) - \left(\left(\frac{\partial V_1}{\partial x}\right)_{M_0} \cos(N_0 x) + \cdots\right) \\ &= \cos(N_0 x)\left[\left(\frac{\partial V_1}{\partial x}\right)_{M_1} - \left(\frac{\partial V_1}{\partial x}\right)_{M_0}\right] \\ &\qquad + \left(\frac{\partial V_1}{\partial x}\right)_{M_1} \{\cos(N_1 x) - \cos(N_0 x)\} + \cdots .\end{aligned}$$

On the basis of the remark about the estimates of the derivatives of V, it follows easily that the quantity in the square brackets is less in absolute

value than a number of the form $aA\delta$. The quantity in braces can be estimated in the following manner:

$$|\cos(N_1 x) - \cos(N_0 x)| = 2\left|\sin\frac{(N_1 x) + (N_0 x)}{2}\right| \cdot \left|\sin\frac{(N_1 x) - (N_0 x)}{2}\right|$$
$$\leq |(N_1 x) - (N_0 x)| \leq (N_1 N_0) < E r_{10}^\lambda = E\delta^\lambda.$$

Hence, the difference of the first two integrals in (24) is less in absolute value than a number of the form

$$3aA\delta + 3aAE\delta^\lambda = A\delta^\lambda(3a\delta^{1-\lambda} + 3aE) < bA\delta^\lambda.$$

It remains to estimate the difference of the two last terms of (24). To this end we consider the subregion (σ) of (S) contained in the interior of the sphere of radius 2δ about M_0. The difference in question can then be written as follows:

$$\int\limits_{(\Sigma-\sigma)} \mu\left\{\frac{\cos(r_{21}N_1)}{r_{21}^2} - \frac{\cos(r_{20}N_0)}{r_{20}^2}\right\} d\sigma$$
$$+ \int\limits_{(\sigma)} \mu\frac{\cos(r_{21}N_1)}{r_{21}^2}\, d\sigma - \int\limits_{(\sigma)} \mu\frac{\cos(r_{20}N_0)}{r_{20}^2}\, d\sigma.$$

From this it follows that

$$\left|\int\limits_{(\Sigma)} \mu\frac{\cos(r_{21}N_1)}{r_{21}^2}\, d\sigma - \int\limits_{(\Sigma)} \mu\frac{\cos(r_{20}N_0)}{r_{20}^2}\, d\sigma\right|$$
$$\leq \left|\int\limits_{(\Sigma-\sigma)} \mu\left\{\frac{\cos(r_{21}N_1)}{r_{21}^2} - \frac{\cos(r_{20}N_0)}{r_{20}^2}\right\} d\sigma\right| \tag{25}$$
$$+ \left|\int\limits_{(\sigma)} \mu\frac{\cos(r_{21}N_1)}{r_{21}^2}\, d\sigma\right| + \left|\int\limits_{(\sigma)} \mu\frac{\cos(r_{20}N_0)}{r_{20}^2}\, d\sigma\right|.$$

According to inequality (22), the last term in (25) is less than $aA(2\delta)^\lambda$. Considering the sphere of radius 3δ about M_1, we can conclude that the next to the last term of (25) is less than $aA(3\delta)^\lambda$. The sum of the last two terms of (25) is hence less than $ca\delta^\lambda$, where c is a constant depending only on the surface (S). We come now to the estimation of the first term on the right-hand side of (25).

We have:

$$\frac{\cos(r_{21}N_1)}{r_{21}^2} - \frac{\cos(r_{20}N_0)}{r_{20}^2}$$
$$= \frac{\cos(r_{21}N_1) - \cos(r_{20}N_0)}{r_{21}^2} + \cos(r_{20}N_0)\left\{\frac{1}{r_{21}^2} - \frac{1}{r_{20}^2}\right\}. \tag{26}$$

We shall first of all consider the second summand of (26). For the sake of simplicity, we assume that $\lambda < 1$. Since outside (σ)

$$\tfrac{1}{2} r_{20} < r_{21} < \tfrac{3}{2} r_{20}$$

it follows that

$$\left| \cos(r_{20} N_0) \left\{ \frac{1}{r_{21}^2} - \frac{1}{r_{20}^2} \right\} \right| < E r_{20}^{\lambda} \frac{|r_{20} - r_{21}| \cdot (r_{20} + r_{21})}{r_{21}^2 r_{20}^2}$$

$$< E r_{20}^{\lambda} \frac{\delta \cdot \frac{5}{2} r_{20}}{\left(\frac{1}{2}\right)^2 r_{20}^4} = b\,\delta \frac{1}{r_{20}^{3-\lambda}}.$$

With the help of cylindrical coordinates we obtain:

$$\left| \int\limits_{(\Sigma-\sigma)} \mu \cos(r_{20} N_0) \left\{ \frac{1}{r_{21}^2} - \frac{1}{r_{20}^2} \right\} d\sigma \right| \leq 2 A b \delta \int_0^{2\pi} d\varphi \int_\delta^d \frac{d\varrho}{\varrho^{2-\lambda}}$$

$$= \frac{4\pi b}{1-\lambda} A \delta \left\{ \frac{1}{\delta^{1-\lambda}} - \frac{1}{d^{1-\lambda}} \right\} < \frac{4\pi b}{1-\lambda} A \delta^{\lambda}.$$

We now turn to the first summand of (26). We have:

$$\begin{aligned} &\cos(r_{21} N_1) - \cos(r_{20} N_0) \\ &\quad = [\cos(r_{21} N_1) - \cos(r_{21} N_0)] + [\cos(r_{21} N_0) - \cos(r_{20} N_0)]. \end{aligned} \tag{27}$$

Clearly, for the first difference on the right-hand side of (27)

$$|\cos(r_{21} N_1) - \cos(r_{21} N_0)| \leq |(r_{21} N_1) - (r_{21} N_0)| \leq (N_1 N_0) < E \delta^{\lambda}.$$

This leads to the following estimate for the corresponding integral:

$$\left| \int\limits_{(\Sigma-\sigma)} \mu \frac{\cos(r_{21} N_1) - \cos(r_{21} N_0)}{r_{21}^2} d\sigma \right| < 8 A E \delta^{\lambda} \int_0^{2\pi} \int_\delta^d \frac{\varrho\, d\varrho\, d\varphi}{r_{20}^2} \tag{28}$$

$$\leq 16\pi E A \delta^{\lambda} \int_\delta^d \frac{d\varrho}{\varrho} = 16\pi E A \delta^{\lambda} \ln \frac{d}{\delta} = a A \delta^{\lambda'}.$$

We now consider the last difference in (27). If z_1 is the z coordinate of the point M_1 in an (x,y,z) coordinate system with origin M_0 and z axis N_0, then

$$\cos(r_{21} N_0) - \cos(r_{20} N_0) = \frac{\zeta - z_1}{r_{21}} - \frac{\zeta}{r_{20}} = \zeta \left(\frac{1}{r_{21}} - \frac{1}{r_{20}} \right) - \frac{z_1}{r_{21}};$$

for the first summand on the right-hand side we have the estimate:

$$|\zeta| \cdot \left| \frac{1}{r_{21}} - \frac{1}{r_{20}} \right| < a r_{20}^{1+\lambda} \frac{\delta}{r_{20} r_{21}} < 2 a \delta \frac{1}{r_{20}^{1-\lambda}} \leq \frac{2 a \delta}{\varrho^{1-\lambda}};$$

for the second summand we have:

$$\left| \frac{z_1}{r_{21}} \right| < \frac{a \delta^{1+\lambda}}{\frac{1}{2} r_{20}} \leq \frac{2 a \delta^{1+\lambda}}{\varrho}.$$

For the integral we then obtain the following estimate:

$$\left| \int\limits_{(\Sigma-\sigma)} \mu \frac{\cos(r_{21}N_0) - \cos(r_{20}N_0)}{r_{21}^2}\, d\sigma \right|$$

$$< 4A \cdot 4\pi \cdot 2a\delta \int\limits_{\delta}^{d} \frac{\varrho\, d\varrho}{\varrho^{3-\lambda}} + 4A \cdot 4\pi \cdot 2a\delta^{1+\lambda} \int\limits_{\delta}^{d} \frac{\varrho\, d\varrho}{\varrho^{3}}$$

$$= \frac{32\pi a}{1-\lambda} A\delta \left\{ \frac{1}{\delta^{1-\lambda}} - \frac{1}{d^{1-\lambda}} \right\} + 32\pi a A \delta^{1+\lambda} \left\{ \frac{1}{\delta} - \frac{1}{d} \right\} < bA\delta^{\lambda}.$$

Collecting all the results, one sees that the difference (23) is less in absolute value than a number of the form

$$cA\delta^{\lambda'};$$

the assertion of the theorem is herewith established. If the density μ is H-continuous then the result obtained can be sharpened; in this case the difference (23) is less than a number of the form $cA\delta^{\lambda}$.

§6. A Theorem on the Normal Derivative of the Potential of the Simple Layer

We assume that the point M_1 lies inside (D_i) or inside (D_e) on the normal to (S) at the point M_0 at a distance δ from M_0.

Theorem. *If the density of the potential of the simple layer is continuous on (S) and if the point M_1 tends to M_0, then*

$$\lim \left(\frac{dV}{dn} \right)_{M_1} = \frac{dV_i}{dn} = \frac{dV}{dn} + 2\pi\mu_0$$

or

$$\lim \left(\frac{dV}{dn} \right)_{M_1} = \frac{dV_e}{dn} = \frac{dV}{dn} - 2\pi\mu_0,$$

according to whether M_1 approaches M_0 from the interior of (D_i) or the interior of (D_e); μ_0 is here the value of μ at the point M_0.

Corollary. *If μ is H-continuous on (S)*, i.e., *if*

$$|\mu - \mu_0| < A r_{20}^{\lambda},$$

then, depending on which is the case in question,

$$\left| \left(\frac{dV}{dn} \right)_{M_1} - \frac{dV}{dn} - 2\pi\mu_0 \right| < aA\delta^{\lambda}$$

or

$$\left| \left(\frac{dV}{dn} \right)_{M_1} - \frac{dV}{dn} + 2\pi\mu_0 \right| < aA\delta^{\lambda}.$$

Proof. We consider the difference

$$\left(\frac{dV}{dn} \right)_{M_1} - \frac{dV}{dn} = \int\limits_{(S)} \mu \frac{\cos(r_{21}N_0)}{r_{21}^2}\, d\sigma - \int\limits_{(S)} \mu \frac{\cos(r_{20}N_0)}{r_{20}^2}\, d\sigma.$$

We have that

$$
\begin{aligned}
&\int\limits_{(S)} \mu \frac{\cos(r_{21} N_0)}{r_{21}^2}\, d\sigma - \int\limits_{(S)} \mu \frac{\cos(r_{20} N_0)}{r_{20}^2}\, d\sigma \\
&\quad = \left(\int\limits_{(S)} \mu \frac{\cos(r_{21} N_2)}{r_{21}^2}\, d\sigma - \int\limits_{(S)} \mu \frac{\cos(r_{20} N_2)}{r_{20}^2}\, d\sigma \right) \\
&\quad + \left(\int\limits_{(S)} \mu \frac{\cos(r_{21} N_0) - \cos(r_{21} N_2)}{r_{21}^2}\, d\sigma - \int\limits_{(S)} \mu \frac{\cos(r_{20} N_0) - \cos(r_{20} N_2)}{r_{20}^2}\, d\sigma \right).
\end{aligned} \tag{29}
$$

The first difference on the right-hand side of (29) has already been studied in §3 as the difference of two values of the potential of a double layer. Under the assumption of the continuity of μ we found that this difference has the limit $2\pi\mu_0$ or $-2\pi\mu_0$ according to whether the point M_1 remains in the interior of (D_i) or in the interior of (D_e); if μ is H-continuous then this difference approximates its limit up to a quantity which is less in absolute value than $aA\delta^\lambda$.

To complete the proof of the theorem it suffices to show that the second difference on the right-hand side of (29) becomes arbitrarily small as $\delta \to 0$. We assume only that μ is bounded and prove that this difference is less in absolute value than a number of the form $aA\delta^\lambda$.

We consider the LYAPUNOV sphere about M_0; let (Σ) be the subregion of (S) lying inside this sphere. The difference in question is then equal to

$$
\begin{aligned}
&\left(\int\limits_{(S-\Sigma)} \mu \frac{\cos(r_{21} N_0) - \cos(r_{21} N_2)}{r_{21}^2}\, d\sigma - \int\limits_{(S-\Sigma)} \mu \frac{\cos(r_{20} N_0) - \cos(r_{20} N_2)}{r_{20}^2}\, d\sigma \right) \\
&+ \left(\int\limits_{(\Sigma)} \mu \frac{\cos(r_{21} N_0) - \cos(r_{21} N_2)}{r_{21}^2}\, d\sigma - \int\limits_{(\Sigma)} \mu \frac{\cos(r_{20} N_0) - \cos(r_{20} N_2)}{r_{20}^2}\, d\sigma \right).
\end{aligned} \tag{30}
$$

If $\delta < \frac{d}{2}$, then the first difference in (30) is less in absolute value than a number of the form $aA\delta$. Indeed, the function

$$
\int\limits_{(S-\Sigma)} \mu \frac{\cos(r_{21} N_0) - \cos(r_{21} N_2)}{r_{21}^2}\, d\sigma
$$

is continuous in a neighborhood of the point M_0 and has there continuous and bounded derivatives. It therefore remains to examine the second difference in (30).

For this purpose we consider the sphere of radius 2δ about M_0 and denote by (σ) the subregion of (S) lying inside this sphere. The difference can then be written in the following manner:

$$\int\limits_{(\Sigma-\sigma)} \mu \left\{ \frac{\cos(r_{21} N_0) - \cos(r_{21} N_2)}{r_{21}^2} - \frac{\cos(r_{20} N_0) - \cos(r_{20} N_2)}{r_{20}^2} \right\} d\sigma$$
$$+ \int\limits_{(\sigma)} \mu \frac{\cos(r_{21} N_0) - \cos(r_{21} N_2)}{r_{21}^2} d\sigma - \int\limits_{(\sigma)} \mu \frac{\cos(r_{20} N_0) - \cos(r_{20} N_2)}{r_{20}^2} d\sigma . \quad (31)$$

Just as in the preceding section,

$$\left.\begin{aligned} |\cos(r_{21} N_0) - \cos(r_{21} N_2)| &\le (N_0 N_2) < E r_{20}^\lambda , \\ |\cos(r_{20} N_0) - \cos(r_{20} N_2)| &\le (N_0 N_2) < E r_{20}^\lambda , \\ |\cos(r_{01} N_0) - \cos(r_{01} N_2)| &\le (N_0 N_2) < E r_{20}^\lambda . \end{aligned}\right\} \quad (32)$$

On the basis of inequality (12) of I, §1, $r_{20} < 2\varrho$ where ϱ is the projection of r_{20} onto the tangent plane at the point M_0. Each of the three absolute values in (32) is therefore less than $2^\lambda E \varrho^\lambda$. Since ϱ is likewise the projection of r_{21} onto the tangent plane, we have $r_{20} \geqq \varrho$ and $r_{21} > \varrho$; hence, each of the last two integrals in (31) is less in absolute value than

$$A \cdot 2^\lambda E \cdot 2 \int_0^{2\pi} \int_0^{2\delta} \frac{\varrho \, d\varrho}{\varrho^{2-\lambda}} d\varphi = 4\pi \cdot 2^\lambda E \cdot A \int_0^{2\delta} \frac{d\varrho}{\varrho^{1-\lambda}} = \frac{4\pi \cdot 4^\lambda E}{\lambda} A \delta^\lambda .$$

It remains now to estimate the first integral of (31). Making use of the transformation used in §3, we find for the point M_1 located anywhere at a distance δ from M_0 that

$$\begin{aligned} T &= \frac{\cos(r_{21} N_0) - \cos(r_{21} N_2)}{r_{21}^2} - \frac{\cos(r_{20} N_0) - \cos(r_{20} N_2)}{r_{20}^2} \\ &= \frac{r_{21}\cos(r_{21} N_0) - r_{21}\cos(r_{21} N_2)}{r_{21}^3} - \frac{r_{20}\cos(r_{20} N_0) - r_{20}\cos(r_{20} N_2)}{r_{20}^3} \\ &= \frac{[r_{21}\cos(r_{21} N_0) - r_{20}\cos(r_{20} N_0)] - [r_{21}\cos(r_{21} N_2) - r_{20}\cos(r_{20} N_2)]}{r_{21}^3} \\ &\quad + [r_{20}\cos(r_{20} N_0) - r_{20}\cos(r_{20} N_2)] \left\{ \frac{1}{r_{21}^3} - \frac{1}{r_{20}^3} \right\} \\ &= \frac{r_{01}\cos(r_{01} N_0) - r_{01}\cos(r_{01} N_2)}{r_{21}^3} \\ &\quad + r_{20}[\cos(r_{20} N_0) - \cos(r_{20} N_2)] \cdot \frac{(r_{20} - r_{21})(r_{20}^2 + r_{20} r_{21} + r_{21}^2)}{r_{20}^3 r_{21}^3} \\ &= \frac{\delta[\cos(r_{01} N_0) - \cos(r_{01} N_2)]}{r_{21}^3} \\ &\quad + [\cos(r_{20} N_0) - \cos(r_{20} N_2)] \cdot (r_{20} - r_{21}) \left(\frac{1}{r_{21}^3} + \frac{1}{r_{20} r_{21}^2} + \frac{1}{r_{20}^2 r_{21}} \right); \end{aligned}$$

the vector from M_1 to M_2 is equal to the sum of the vectors from M_1 to M_0 and from M_0 to M_2.

If we consider the second and third of inequalities (32) and recall that on $(\Sigma - \sigma)$ the inequality $r_{21} > \frac{r_{20}}{2}$ holds, then we obtain:

$$|T| < \frac{\delta E r_{20}^{\lambda}}{(\frac{1}{2} r_{20})^3} + \frac{14 E r_{20}^{\lambda} \delta}{r_{20}^3} = c \frac{\delta}{r_{20}^{3-\lambda}} \leq c \frac{\delta}{\varrho^{3-\lambda}}.$$

For the first integral in (31) we then have the estimate:

$$\left| \int\limits_{(\Sigma-\sigma)} \mu T \, d\sigma \right| < cA \cdot 2\delta \int\limits_0^{2\pi} d\varphi \int\limits_{\delta}^{a} \frac{\varrho \, d\varrho}{\varrho^{3-\lambda}} = \frac{4\pi c}{1-\lambda} A \delta \left\{ \frac{1}{\delta^{1-\lambda}} - \frac{1}{d^{1-\lambda}} \right\} < b A \delta^{\lambda}.$$

The inequalities obtained complete the proof of the theorem.

§7. On the Derivatives of the Potential of the Simple Layer

We first give an example of a potential of a simple layer with continuous density which possesses unbounded derivatives.

Let the circle of radius $R < 1$ about the origin in the (x,y) plane be a subregion of a closed LYAPUNOV surface (S). We then consider the potential of a simple layer with density μ on the surface (S). The behavior of this potential in a sufficiently small neighborhood of the origin depends then on the behavior of the integral

$$\iint\limits_{\xi^2+\eta^2 \leq R^2} \frac{\mu(\xi, \eta)}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}} \, d\xi \, d\eta$$

in the same neighborhood, since the integral over the remaining part of (S) possesses continuous derivatives of arbitrary order in a neighborhood of the origin.

Inside the circle mentioned we put

$$\mu(\xi, \eta) = \frac{\xi}{\sqrt{\xi^2+\eta^2} \, |\ln \sqrt{\xi^2+\eta^2}|};$$

$\mu(\xi,\eta)$ is continuous and has limit zero as $\sqrt{\xi^2+\eta^2} \to 0$.

Computing the derivative of the integral with respect to x at the point $M_1(0, 0, z)$, $|z| > 0$, we obtain:

$$\iint\limits_{\xi^2+\eta^2 \leq R^2} \frac{\xi \cdot \xi \, d\xi \, d\eta}{\sqrt{\xi^2+\eta^2} \, |\ln \sqrt{\xi^2+\eta^2}| \, (\sqrt{\xi^2+\eta^2+z^2})^3}$$

$$= \int\limits_0^{2\pi} \cos^2\varphi \, d\varphi \int\limits_0^{R} \frac{\varrho^2 \, d\varrho}{|\ln \varrho| \, (\sqrt{\varrho^2+z^2})^3} = \pi \int\limits_0^{R} \frac{\varrho^2 \, d\varrho}{|\ln \varrho| \, (\sqrt{\varrho^2+z^2})^3}.$$

The last integral increases as $|z|$ decreases. We wish to show that it increases

without bound. Let δ be an arbitrary number in the interval $0 < \delta < R$; then

$$\int_0^R \frac{\varrho^2\, d\varrho}{|\ln \varrho|\,(\sqrt{\varrho^2 + z^2})^3} > \int_\delta^R \frac{\varrho^2\, d\varrho}{|\ln \varrho|\,(\sqrt{\varrho^2 + z^2})^3}\,.$$

But as $|z| \to 0$,

$$\int_\delta^R \frac{\varrho^2\, d\varrho}{|\ln \varrho|\,(\sqrt{\varrho^2 + z^2})^3} \to \int_\delta^R \frac{d\varrho}{\varrho\,|\ln \varrho|} = \ln \left|\frac{\ln \delta}{\ln R}\right|.$$

Hence, for sufficiently small $|z|$ the integral becomes greater than $\frac{1}{2} \ln \left|\frac{\ln \delta}{\ln R}\right|$ so that with the same value of $|z|$

$$\int_0^R \frac{\varrho^2\, d\varrho}{|\ln \varrho|\,(\sqrt{\varrho^2 + z^2})^3} > \frac{1}{2} \ln \left|\frac{\ln \delta}{\ln R}\right|.$$

Since δ was arbitrary, it follows that the integral on the left-hand side increases without bound. It is herewith shown that the derivative with respect to x of the potential of a simple layer with continuous density considered is unbounded.

The following theorem is due to LYAPUNOV.

Theorem (LYAPUNOV). *If the density μ of the potential of a simple layer is H-continuous on (S), then the derivatives*

$$\left.\begin{aligned} \frac{\partial V}{\partial x} &= \int_{(S)} \mu \frac{\xi - x_1}{r_{21}^3}\, d\sigma\,, \\ \frac{\partial V}{\partial y} &= \int_{(S)} \mu \frac{\eta - y_1}{r_{21}^3}\, d\sigma\,, \\ \frac{\partial V}{\partial z} &= \int_{(S)} \mu \frac{\zeta - z_1}{r_{21}^3}\, d\sigma\,, \end{aligned}\right\} \tag{33}$$

considered as functions of the point M_1 (x_1, y_1, z_1) are H-continuous in the interior of (D_i) and in the interior of (D_e).

The somewhat complicated proof of this theorem can be found in Appendix I. Assuming the theorem true, we wish here to draw certain conclusions.

From the first theorem of I, §2 and the continuity considerations which follow it we conclude that the derivatives (33) possess definite limits as the point M_1 approaches a point of M_0 of (S) and that these limits themselves constitute an H-continuous function of the point M_0. We denote the limits by

$$\frac{\partial V_i}{\partial x}, \quad \frac{\partial V_i}{\partial y}, \quad \frac{\partial V_i}{\partial z}, \quad \frac{\partial V_e}{\partial x}, \quad \frac{\partial V_e}{\partial y}, \quad \frac{\partial V_e}{\partial z},$$

where the indices i and e indicate the region in which the point M_1 lies.

The derivatives have jumps as the point M_1 passes through the surface (S). The theorem in I, §3 enables us to determine the values of these jumps. Indeed, if $\frac{\partial V_i}{\partial x}$ and $\frac{\partial V_e}{\partial x}$ are the derivatives of two functions one of which is defined in (D_i) and the other in (D_e) and if these functions coincide on (S), then from the theorem mentioned it follows that

$$\frac{\partial V_i}{\partial x} - \frac{\partial V_e}{\partial x} = \cos(N_0 x)\left(\frac{dV_i}{dn} - \frac{dV_e}{dn}\right) = 4\pi\mu_0\cos(N_0 x).$$

We have similarly that

$$\frac{\partial V_i}{\partial y} - \frac{\partial V_e}{\partial y} = 4\pi\mu_0\cos(N_0 y),$$

$$\frac{\partial V_i}{\partial z} - \frac{\partial V_e}{\partial z} = 4\pi\mu_0\cos(N_0 z).$$

§8. The Derivatives of the Potential of a Simple Layer with Differentiable Density

We assume that μ considered as a function of the point $M_2(\xi,\eta,\zeta)$ of (S) admits the quantities

$$D_\xi\mu, \; D_\eta\mu, \; D_\zeta\mu.$$

We suppose moreover that the surface (S) has continuous curvature, i.e., the quantities

$$D_\xi\cos(N_2 x), \; D_\eta\cos(N_2 y), \; D_\zeta\cos(N_2 z)$$

are defined and continuous.

We consider the integral

$$V = \int\limits_{(\Sigma)} \mu \frac{d\sigma}{r_{20}},$$

where (Σ) is a certain subregion of (S) bounded by the curve (l).

If the point $M_0(x,y,z)$ does not lie on (Σ), then

$$\begin{aligned}\frac{\partial V}{\partial x} &= \int_{(\Sigma)} \mu \frac{\partial \frac{1}{r_{20}}}{\partial x} d\sigma = -\int_{(\Sigma)} \mu \frac{\partial \frac{1}{r_{20}}}{\partial \xi} d\sigma \\ &= -\int_{(\Sigma)} \mu D_\xi \frac{1}{r_{20}} d\sigma + \int_{(\Sigma)} \mu \cos(N_2 x) \frac{\cos(r_{20} N_2)}{r_{20}^2} d\sigma;\end{aligned} \tag{34}$$

indeed,

$$D_\xi \frac{1}{r_{20}} = \frac{\partial \frac{1}{r_{20}}}{\partial \xi} - \cos(N_2 x) \frac{d \frac{1}{r_{20}}}{d n_2} = \frac{\partial \frac{1}{r_{20}}}{\partial \xi} + \cos(N_2 x) \frac{\cos(r_{20} N_2)}{r_{20}^2} \,.$$

We do not change the form of the last integral in (34) further. The integrands of the first integral, on the other hand, we write in the form

$$\mu D_\xi \frac{1}{r_{20}} = D_\xi \left(\frac{\mu}{r_{20}}\right) - \frac{1}{r_{20}} D_\xi \mu \,.$$

We now have

$$\frac{\partial V}{\partial x} = \int_{(\Sigma)} \mu \cos(N_2 x) \frac{\cos(r_{20} N_2)}{r_{20}^2} d\sigma + \int_{(\Sigma)} \frac{D_\xi \mu}{r_{20}} d\sigma - \int_{(\Sigma)} D_\xi \left(\frac{\mu}{r_{20}}\right) d\sigma \,. \tag{35}$$

We make an additional transformation of the last integral. For brevity we introduce the following notation:

$$\mu \frac{1}{r_{20}} = f(\xi, \eta, \zeta) = f;$$

$$\cos(N_2 x) = \alpha \,, \qquad \cos(N_2 y) = \beta \,, \qquad \cos(N_2 z) = \gamma \,.$$

Considering that

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

and hence that

$$\tfrac{1}{2} D_\xi (\alpha^2 + \beta^2 + \gamma^2) = \alpha D_\xi \alpha + \beta D_\xi \beta + \gamma D_\xi \gamma = 0 \,,$$

we obtain

$$D_\xi f = D_\xi[\alpha(\alpha f) + \beta(\beta f) + \gamma(\gamma f)] = \alpha D_\xi(f\alpha) + \beta D_\xi(f\beta) + \gamma D_\xi(f\gamma) \,.$$

In order to apply STOKES' formula, we put

$$\varphi = 0, \quad \psi = f\gamma, \quad \chi = -f\beta \,.$$

We then have

$$\begin{aligned}\alpha D_\xi(f\alpha) + \beta D_\xi(f\beta) + \gamma D_\xi(f\gamma) &= \alpha[D_\eta \chi - D_\zeta \psi] + \beta[D_\zeta \varphi - D_\xi \chi] \\ &+ \gamma[D_\xi \psi - D_\eta \varphi] + \alpha\{D_\xi(\alpha f) + D_\eta(\beta f) + D_\zeta(\gamma f)\} \,.\end{aligned}$$

Applying STOKES' formula, one obtains

$$\int_{(\Sigma)} D_\xi\left(\frac{\mu}{r_{20}}\right) d\sigma = \int_{(l)} \mu \frac{\cos(Nz)\, d\eta - \cos(Ny)\, d\zeta}{r_{20}} + \int_{(\Sigma)} \alpha \{D_\xi(\alpha f) + D_\eta(\beta f) + D_\zeta(\gamma f)\}\, d\sigma . \tag{36}$$

We introduce the following notation

$$K = D_\xi \alpha + D_\eta \beta + D_\zeta \gamma$$

It was pointed out in I, §3 that K is equal to the mean curvature of the surface.

From I, §3 it follows that for any arbitrary function f

$$\alpha D_\xi f + \beta D_\eta f + \gamma D_\zeta f = 0$$

The sum inside the braces of the last integral of (36) therefore assumes the following form:

$$D_\xi(\alpha f) + D_\eta(\beta f) + D_\zeta(\gamma f) = (\alpha D_\xi f + \beta D_\eta f + \gamma D_\xi f) + f(D_\xi \alpha + D_\eta \beta + D_\zeta \gamma) = fK = \frac{\mu}{r_{20}} K . \tag{37}$$

Combining (35), (36), and (37), we obtain:

$$\frac{\partial V}{\partial x} = \int_{(\Sigma)} [D_\xi \mu - \mu K \cos(Nx)] \frac{d\sigma}{r_{20}} + \int_{(\Sigma)} \mu \cos(Nx) \frac{\cos(r_{20} N)}{r_{20}^2}\, d\sigma + \int_{(l)} \mu \frac{\cos(Ny)\, d\zeta - \cos(Nz)\, d\eta}{r_{20}} . \tag{38}$$

The derivatives $\dfrac{\partial V}{\partial y}$ and $\dfrac{\partial V}{\partial z}$ are obtained by cyclic permutation of the letters x, y, z and ξ, η, ζ. If the surface is closed then the last term in (38) does not appear.

§9. The Normal Derivative of the Potential of a Double Layer

In order that the normal derivative of the potential of a double layer should exist, the density μ of this layer must satisfy more stringent conditions than any we have met so far.

The following example, which goes back to LYAPUNOV, shows that the H-continuity of the density is not sufficient for the existence of a derivative. Let (Σ) be a planar subregion of (S); to be specific, suppose (Σ) is a disk with radius d with the point M_0 as origin. We put

$$\mu = K\varrho \qquad (K > 0)$$

on (Σ), where ϱ is the distance of a point of the subregion (Σ) from M_0.

If M_1 and M_2 are two points of (Σ) a distance r apart, then

$$|\mu_{M_1} - \mu_{M_2}| = K\,|\varrho_2 - \varrho_1| \leq K r.$$

The function μ is therefore H-continuous on (Σ) with exponent λ equal to one.

We have:
$$W = \int\limits_{(\Sigma)} \mu \frac{\cos(rN)}{r^2}\,d\sigma + \int\limits_{(S-\Sigma)} \mu \frac{\cos(rN)}{r^2}\,d\sigma.$$

Since the second term possesses derivatives of arbitrary order in a neighborhood of the point M_0, we concern ourselves only with the first integral.

We choose the normal to (Σ) at the point M_0 as negative z axis. For a point $M_1(0, 0, z)$ on the normal to (Σ) at the point M_0

$$r^2 = \xi^2 + \eta^2 + z^2 = \varrho^2 + z^2, \qquad \cos(rN) = \frac{z}{r}$$

and

$$W_1 = \int\limits_{(\Sigma)} \mu \frac{\cos(rN)}{r^2}\,d\sigma = Kz \int\limits_0^{2\pi} d\varphi \int\limits_0^d \frac{\varrho^2\,d\varrho}{(\sqrt{\varrho^2 + z^2})^3}$$
$$= 2\pi K z \int\limits_0^d \frac{d\varrho}{\sqrt{\varrho^2 + z^2}} - 2\pi K z^3 \int\limits_0^d \frac{d\varrho}{(\sqrt{\varrho^2 + z^2})^3}$$
$$= 2\pi K z \ln\left(d + \sqrt{d^2 + z^2}\right) - 2\pi K z \ln|z| - 2\pi K z \frac{d}{\sqrt{d^2 + z^2}}.$$

The first and third terms of W_1 possess derivatives with respect to z which have finite limits as $z \to 0$; the derivative of the second summand is equal to

$$-2\pi K \ln|z| - 2\pi K$$

and increases without bound as z tends to zero.

We shall subsequently make no use of normal derivatives of the potential of a double layer. We present here two theorems without proof; they are due to LYAPUNOV, but we have formulated them somewhat more generally; the proof of these theorems may be found in Appendix II.

Theorem 1. *Let one of the following two conditions be satisfied*:

1. *μ is continuous on (S) and $(N_1 N_2) < E r_{21}$ (N_1 and N_2 are the normals to (S) at the points M_1 and M_2 a distance r_{21} apart), or*

2. $$|\mu_{M_1} - \mu_{M_2}| < E r_{21}, \quad (N_1 N_2) < E r_{21}^{\lambda}, \quad (0 < \lambda \leq 1).$$

If then the potential of a double layer

$$W = \int\limits_{(S)} \mu \frac{\cos(r_{20} N)}{r_{20}^2}\,d\sigma$$

possesses one of the derivatives

$$\frac{dW_i}{dn}, \quad \frac{dW_e}{dn},$$

it also possesses the other, and moreover

$$\frac{dW_i}{dn} = \frac{dW_e}{dn}.$$

We take the point M_0 of the surface as origin of a system of cylindrical coordinates in which the z axis coincides with the normal at M_0. Let (Σ) be the subregion of (S) contained in the interior of the LYAPUNOV sphere about M_0. As we know, (Σ) is cut at most once by any line parallel to the z axis. Let now a function $\mu(M)$ be given on (Σ) where M is a point of (Σ). We put

$$\mu(\varrho, \varphi) = \mu(M)$$

(ϱ, φ are the first two coordinates of M), and we denote the value of μ at M_0 by μ_0. The inequality

$$\left| \int_0^{2\pi} [\mu(\varrho, \varphi) - \mu_0] \, d\varphi \right| < a \varrho^{1+\nu} \qquad (a > 0, \nu > 0)$$

is called a LYAPUNOV *condition.*

Theorem 2. *If one of the two conditions of Theorem 1 and also a* LYAPUNOV *condition are satisfied, then the potential of the double layer possesses normal derivatives.*

§10. The Derivatives of the Potential of a Double Layer with Differentiable Density

Let (Σ) be a subregion of a surface (S) satisfying the LYAPUNOV conditions, and let (l) be its boundary curve. We consider the integral

$$W = \int_{(\Sigma)} \mu \frac{\cos(r_{20} N_2)}{r_{20}^2} \, d\sigma .$$

The density μ is assumed to be differentiable on (S); this ensures existence of the quantities

$$D_\xi \mu, \; D_\eta \mu, \; D_\zeta \mu.$$

If the point $M_0(x,y,z)$ does not lie on (Σ), then

$$W = \int_{(\Sigma)} \mu \left\{ \frac{\partial \frac{1}{r_{20}}}{\partial x} \cos(Nx) + \frac{\partial \frac{1}{r_{20}}}{\partial y} \cos(Ny) + \frac{\partial \frac{1}{r_{20}}}{\partial z} \cos(Nz) \right\} d\sigma;$$

from this it follows that

$$\begin{aligned}\frac{\partial W}{\partial x} &= \int\limits_{(\Sigma)} \mu \left\{ \frac{\partial^2 \frac{1}{r_{20}}}{\partial x^2} \cos(Nx) + \frac{\partial^2 \frac{1}{r_{20}}}{\partial x \partial y} \cos(Ny) + \frac{\partial^2 \frac{1}{r_{20}}}{\partial x \partial z} \cos(Nz) \right\} d\sigma \\ &= \int\limits_{(\Sigma)} \mu \left\{ \frac{\partial^2 \frac{1}{r_{20}}}{\partial \xi^2} \cos(Nx) + \frac{\partial^2 \frac{1}{r_{20}}}{\partial \xi \partial \eta} \cos(Ny) + \frac{\partial^2 \frac{1}{r_{20}}}{\partial \xi \partial \zeta} \cos(Nz) \right\} d\sigma .\end{aligned} \tag{39}$$

For convenience of notation we introduce the symbols

$$\frac{1}{r_{20}} = f(\xi, \eta, \zeta) = f,$$

$$\cos(Nx) = \alpha, \qquad \cos(Ny) = \beta, \qquad \cos(Nz) = \gamma.$$

We have:

$$\frac{\partial^2 f}{\partial \xi^2} + \frac{\partial^2 f}{\partial \eta^2} + \frac{\partial^2 f}{\partial \zeta^2} = 0 .$$

From the definition of D_ξ, D_η, and D_ζ the equations

$$\beta \frac{\partial^2 f}{\partial \xi \partial \eta} - \alpha \frac{\partial^2 f}{\partial \eta^2} = \beta D_\xi \frac{\partial f}{\partial \eta} - \alpha D_\eta \frac{\partial f}{\partial \eta}$$

$$\gamma \frac{\partial^2 f}{\partial \xi \partial \zeta} - \alpha \frac{\partial^2 f}{\partial \zeta^2} = \gamma D_\xi \frac{\partial f}{\partial \zeta} - \alpha D_\zeta \frac{\partial f}{\partial \zeta}$$

hold for any function f. The quantity inside the braces in the last integral of equation (39) may therefore be written in the following manner:

$$\begin{aligned}&\alpha \frac{\partial^2 f}{\partial \xi^2} + \beta \frac{\partial^2 f}{\partial \xi \partial \eta} + \gamma \frac{\partial^2 f}{\partial \xi \partial \zeta} \\ &\qquad = \left[\beta D_\xi \frac{\partial f}{\partial \eta} + \gamma D_\xi \frac{\partial f}{\partial \zeta}\right] - \alpha \left[D_\eta \frac{\partial f}{\partial \eta} + D_\zeta \frac{\partial f}{\partial \zeta}\right].\end{aligned}$$

The integrand in (39) differs from this expression by a factor μ. Making use of the equation

$$\mu D_\xi \frac{\partial f}{\partial \eta} = D_\xi\left(\mu \frac{\partial f}{\partial \eta}\right) - \frac{\partial f}{\partial \eta} D_\xi \mu$$

and the analogous relations, we may write the integrands in question in the following form:

$$\begin{aligned}&\left\{\alpha \left[D_\eta\left(-\mu \frac{\partial f}{\partial \eta}\right) - D_\zeta\left(\mu \frac{\partial f}{\partial \zeta}\right)\right] + \beta \left[D_\zeta 0 - D_\xi\left(-\mu \frac{\partial f}{\partial \eta}\right)\right]\right. \\ &\left. + \gamma \left[D_\xi\left(\mu \frac{\partial f}{\partial \zeta}\right) - D_\eta 0\right]\right\} + \frac{\partial f}{\partial \eta}(\alpha D_\eta \mu - \beta D_\xi \mu) + \frac{\partial f}{\partial \zeta}(\alpha D_\zeta \mu - \gamma D_\xi \mu).\end{aligned} \tag{40}$$

Using STOKES' formula, the integral over (Σ) of the expression in the braces may be transformed into an integral over the boundary curve (l); one obtains:

$$\int\limits_{(l)}\left(\mu\frac{\partial f}{\partial\zeta}\,d\eta-\mu\frac{\partial f}{\partial\eta}\,d\zeta\right)=\int\limits_{(l)}\mu\left\{\frac{\partial\frac{1}{r_{20}}}{\partial\zeta}\,d\eta-\frac{\partial\frac{1}{r_{20}}}{\partial\eta}\,d\zeta\right\}.$$

Since

$$\frac{\partial f}{\partial\eta}=\frac{\partial\frac{1}{r_{20}}}{\partial\eta}=-\frac{\partial\frac{1}{r_{20}}}{\partial y}$$

we find that

$$\int\limits_{(\Sigma)}\frac{\partial f}{\partial\eta}(\alpha D_\eta\mu-\beta D_\xi\mu)\,d\sigma=-\frac{\partial}{\partial y}\int\limits_{(\Sigma)}\frac{\cos(Nx)\,D_\eta\mu-\cos(Ny)\,D_\xi\mu}{r_{20}}\,d\sigma.$$

One obtains an analogous expression for the integral of

$$\frac{\partial f}{\partial\zeta}(\alpha D_\zeta\mu-\gamma D_\xi\mu).$$

As a result of integrating (40) over (Σ), we therefore obtain:

$$\begin{aligned}\frac{\partial W}{\partial x}&=\frac{\partial}{\partial y}\int\limits_{(\Sigma)}\frac{\cos(Ny)\,D_\xi\mu-\cos(Nx)\,D_\eta\mu}{r_{20}}\,d\sigma\\&+\frac{\partial}{\partial z}\int\limits_{(\Sigma)}\frac{\cos(Nz)\,D_\xi\mu-\cos(Nx)\,D_\zeta\mu}{r_{20}}\,d\sigma\\&+\int\limits_{(l)}\mu\left\{\frac{\partial\frac{1}{r_{20}}}{\partial\zeta}\,d\eta-\frac{\partial\frac{1}{r_{20}}}{\partial\eta}\,d\zeta\right\}.\end{aligned}\tag{41}$$

The formulas for $\dfrac{\partial W}{\partial y}$ and $\dfrac{\partial W}{\partial z}$ are found by cyclic permutation of x, y, z and ξ, η, ζ.

If (S) is a closed surface, then the line integral does not appear, i.e., we then have:

$$\begin{aligned}\frac{\partial W}{\partial x}&=\frac{\partial}{\partial y}\int\limits_{(S)}\frac{\cos(Ny)\,D_\xi u-\cos(Nx)\,D_\eta\mu}{r_{20}}\,d\sigma\\&+\frac{\partial}{\partial z}\int\limits_{(S)}\frac{\cos(Nz)\,D_\xi\mu-\cos(Nx)\,D_\zeta\mu}{r_{20}}\,d\sigma.\end{aligned}\tag{42}$$

We assume that the functions $D_\xi\mu$, $D_\eta\mu$, and $D_\zeta\mu$ are H-continuous on (S). Then $\cos(Ny)D_\xi\mu-\cos(Nx)D_\eta\mu$ is likewise H-continuous; hence, the

potential of the simple layer with this density has H-continuous derivatives in (D_i) and (D_e). From (42) it follows therefore that the derivatives of the potential of a double layer with density μ for which $D_\xi\mu$, $D_\eta\mu$, and $D_\zeta\mu$ are H-continuous functions have H-continuous first derivatives in (D_i) and in (D_e). Making use of the facts that

$$\frac{\partial f}{\eta}(\alpha D_\eta \mu - \beta D_\xi \mu) + \frac{\partial f}{\partial \zeta}(\alpha D_\zeta \mu - \gamma D_\xi \mu)$$
$$= \alpha\left[\frac{\partial f}{\partial \xi} D_\xi \mu + \frac{\partial f}{\partial \eta} D_\eta \mu + \frac{\partial f}{\partial \zeta} D_\zeta \mu\right] - \left[\alpha \frac{\partial f}{\partial \xi} + \beta \frac{\partial f}{\partial \eta} + \gamma \frac{\partial f}{\partial \zeta}\right] D_\xi \mu$$

and

$$\alpha \frac{\partial f}{\partial \xi} + \beta \frac{\partial f}{\partial \eta} + \gamma \frac{\partial f}{\partial \zeta} = -\frac{\cos(r_{20} N)}{r_{20}^2}$$

we obtain in place of formula (42):

$$\begin{aligned} \frac{\partial W}{\partial x} = &- \frac{\partial}{\partial x}\int_{(S)} \frac{\cos(Nx) D_\xi \mu}{r_{20}} d\sigma - \frac{\partial}{\partial y}\int_{(S)} \frac{\cos(Nx) D_\eta \mu}{r_{20}} d\sigma \\ &- \frac{\partial}{\partial z}\int_{(S)} \frac{\cos(Nx) D_\zeta \mu}{r_{20}} d\sigma + \int_{(S)} D_\xi \mu \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma . \end{aligned} \tag{43}$$

Recalling the formulas of §7, we find:

$$\begin{aligned} \frac{\partial W_i}{\partial x} - \frac{\partial W_e}{\partial x} = &- 4\pi \cos^2(Nx) D_x \mu - 4\pi \cos(Nx)\cos(Ny) D_y \mu \\ &- 4\pi \cos(Nx)\cos(Nz) D_z \mu + 4\pi D_x \mu . \end{aligned}$$

Since

$$\cos(Nx) D_x \mu + \cos(Ny) D_y \mu + \cos(Nz) D_z \mu = 0$$

we finally obtain:

$$\frac{\partial W_i}{\partial x} - \frac{\partial W_e}{\partial x} = 4\pi D_x \mu .$$

In a similar manner, we find that

$$\frac{\partial W_i}{\partial y} - \frac{\partial W_e}{\partial y} = 4\pi D_y \mu , \qquad \frac{\partial W_i}{\partial z} - \frac{\partial W_e}{\partial z} = 4\pi D_z \mu .$$

From the last three formulas it follows immediately that

$$\frac{dW_i}{dn} - \frac{dW_e}{dn} = 4\pi[\cos(Nx) D_x \mu + \cos(Ny) D_y \mu + \cos(Nz) D_z \mu] = 0 .$$

§11. On the Convergence of Certain Integrals

We wish to prove the convergence of the following integrals, which are related to the Newtonian potential to be introduced in the next section:

$$\int_{(D)} \frac{d\tau}{r_{20}}, \quad \int_{(D)} \frac{d\tau}{r_{20}^2}, \quad \int_{(D)} \frac{|\mu-\mu_0|}{r_{20}^3}\, d\tau \quad \text{with} \quad |\mu-\mu_0| < A r_{20}^{\lambda}.$$

The integrals are extended over a finite region (D); r_{20} denotes the distance of the point $M_0(x,y,z)$ from the point of integration $M_2(\xi,\eta,\zeta)$.

We describe two spheres of radii δ_1 and $\delta_2 (\delta_1 < \delta_2)$ about the point M_0 and let (δ) be the region bounded by these spheres. To establish the convergence of the integrals in question, it suffices to show that the absolute values of the corresponding integrals over (δ) become arbitrarily small as $\delta_2 \to 0$.

We introduce polar coordinates with the point M_0 as pole:

$$\begin{aligned} \xi - x_0 &= \varrho \sin\Theta \cos\varphi, \\ \eta - y_0 &= \eta \sin\Theta \sin\varphi, \\ \zeta - z_0 &= \varrho \cos\Theta \qquad (0 \leq \Theta \leq \pi,\ 0 \leq \varphi < 2\pi). \end{aligned}$$

We then find:

$$\left.\begin{aligned} \int_{(\delta)} \frac{d\tau}{r_{20}} &= \int_0^{2\pi}\int_0^{\pi}\int_{\delta_1}^{\delta_2} \frac{\varrho^2 \sin\Theta\, d\varrho\, d\Theta\, d\varphi}{\varrho} = 4\pi \cdot \frac{1}{2}(\delta_2^2 - \delta_1^2) < 2\pi\delta_2^2, \\ \int_{(\delta)} \frac{d\tau}{r_{20}^2} &= \int_0^{2\pi}\int_0^{\pi}\int_{\delta_1}^{\delta_2} \frac{\varrho^2 \sin\Theta\, d\varrho\, d\Theta\, d\varphi}{\varrho^2} = 4\pi(\delta_2 - \delta_1) < 4\pi\delta_2, \\ \int_{(\delta)} \frac{|\mu-\mu_0|}{r_{20}^3}\, d\tau &< A\int_0^{2\pi}\int_0^{\pi}\int_{\delta_1}^{\delta_2} \frac{\varrho^2 \sin\Theta\, d\varrho\, d\Theta\, d\varphi}{\varrho^{3-\lambda}} = \frac{4\pi A}{\lambda}(\delta_2^{\lambda} - \delta_1^{\lambda}) < \frac{4\pi}{\lambda} A\,\delta_2^{\lambda}. \end{aligned}\right\} \quad (44)$$

The convergence of the integrals is herewith established. Denoting the sphere about M_0 with radius δ_2 by (δ_2), we have:

$$\left.\begin{aligned} \int_{(\delta_2)} \frac{d\tau}{r_{20}} &= 2\pi\delta_2^2, \\ \int_{(\delta_2)} \frac{d\tau}{r_{20}^2} &= 4\pi\delta_2, \\ \int_{(\delta_2)} \frac{|\mu-\mu_0|}{r_{20}^3}\, d\tau &< aA\delta_2^{\lambda}. \end{aligned}\right\} \quad (45)$$

§12. On the Newtonian Potential

Let (D) be a finite region and μ an integrable function in this region. The integral

$$P = \int\limits_{(D)} \frac{\mu\, d\tau}{r_{20}}, \tag{46}$$

considered as a function of the point $M_0(x,y,z)$ is called the *Newtonian potential.* It is clear that at any point lying outside (D) P possesses all derivatives with respect to x, y, and z. If R denotes the distance of the point M_0 from a certain fixed point of space, then with increasing R the function P goes to zero as the first power of $\frac{1}{R}$. Each of the derivatives $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, and $\frac{\partial P}{\partial z}$ go to zero with increasing R as the second power of $\frac{1}{R}$. Moreover, at any point not belonging to either (D) or the boundary of (D)

$$\Delta P = \int\limits_{(D)} \mu \cdot \Delta \frac{1}{r_{20}}\, d\tau = 0,$$

i.e., P is a harmonic function in any region which together with its boundary lies outside of (D). The first of inequalities (44) shows that if the density μ is bounded and integrable the integral (46) converges. Indeed, making use of the notation of §11, we obtain:

$$\left| \int\limits_{(\delta)} \mu \frac{d\tau}{r_{20}} \right| < A \int\limits_{(\delta)} \frac{d\tau}{r_{20}} \leq 2\pi A \delta_2^2 \to 0 \qquad (\delta_2 \to 0).$$

We wish to compute the value of P for the case in which the region (D) is a sphere and the density μ is a constant; we put $\mu = 1$.

It is clear that in this case P depends only on the distance a of the point M_0 from the center of the sphere and not on the direction of the line segment joining these two points. Making use of this fact, we choose the center of the sphere as origin of our coordinate system such that the z axis passes through the point M_0.

We introduce the polar coordinates

$$\xi = \varrho \sin \Theta \cos \varphi,$$
$$\eta = \varrho \sin \Theta \sin \varphi,$$
$$\zeta = \varrho \cos \Theta.$$

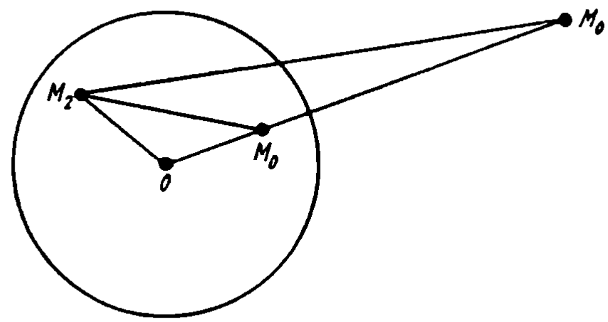

Fig. 21

In this coordinate system then Θ is the angle between OM_2 and OM_0 (Fig. 21); we obtain:

$$r_{20} = \sqrt{\varrho^2 + a^2 - 2a\varrho\cos\Theta}. \tag{47}$$

If R is the radius of the sphere, then

$$P = \int_{(D)} \frac{d\tau}{r_{20}} = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \frac{\varrho^2 \sin\Theta \, d\varrho \, d\Theta \, d\varphi}{r_{20}}.$$

We choose $(\varrho, r_{20}, \varphi)$ as integration variables. Since

$$\frac{\partial(\varrho, r_{20}, \varphi)}{\partial(\varrho, \Theta, \varphi)} = \frac{\varrho a \sin\Theta}{r_{20}}, \qquad \frac{\partial(\varrho, \Theta, \varphi)}{\partial(\varrho, r_{20}, \varphi)} = \frac{r_{20}}{\varrho a \sin\Theta}$$

We obtain:

$$P = \frac{1}{a}\int_0^R \int_{|a-\varrho|}^{a+\varrho} \int_0^{2\pi} \varrho \, d\varphi \, dr_{20} \, d\varrho = \frac{2\pi}{a}\int_0^R \varrho\,[(a+\varrho) - |a-\varrho|]\,d\varrho.$$

If M_0 lies outside the sphere (D), then $a > \varrho$ and

$$P = \frac{2\pi}{a}\int_0^R \varrho\,[(a+\varrho) - (a-\varrho)]\,d\varrho = \frac{2\pi}{a}\int_0^R 2\varrho^2 d\varrho = \frac{4\pi}{a}\cdot\frac{R^3}{3} = \frac{4\pi R^3}{3}\cdot\frac{1}{a}.$$

If M_0 lies inside the sphere, then $|a - \varrho| = a - \varrho$ if $\varrho < a$ and $|a - \varrho| = \varrho - a$ if $\varrho > a$. Hence,

$$P = \frac{2\pi}{a}\left\{\int_0^a \varrho\,[(a+\varrho) - (a-\varrho)]\,d\varrho + \int_a^R \varrho\,[(a+\varrho) - (\varrho - a)]\,d\varrho\right\}$$

$$= \frac{2\pi}{a}\left[\int_0^a 2\varrho^2 d\varrho + 2a\int_a^R \varrho\, d\varrho\right] = \frac{2\pi}{a}\left[\frac{2}{3}a^3 + 2a\left(\frac{R^2}{2} - \frac{a^2}{2}\right)\right]$$

$$= \frac{2\pi}{a}\left[aR^2 - \frac{a^3}{3}\right] = 2\pi\left(R^2 - \frac{a^2}{3}\right).$$

We have therefore that

$$\left.\begin{aligned} P &= \frac{4\pi R^3}{3}\cdot\frac{1}{a}, && \text{for } M_0 \text{ in } (D_e) \text{ and} \\ P &= 2\pi R^2 - \frac{2\pi}{3}a^2, && \text{for } M_0 \text{ in } (D_i)\,. \end{aligned}\right\} \tag{48}$$

From the formulas (48) one easily sees that the potential and its first derivatives are continuous everywhere. As we shall see in the next section, the Newtonian potential of an arbitrary finite region possesses this property if the density μ is bounded and integrable.

We now wish to derive an approximation formula for the value of the Newtonian potential of an arbitrary finite region (D) with bounded density μ at a point a great distance from (D).

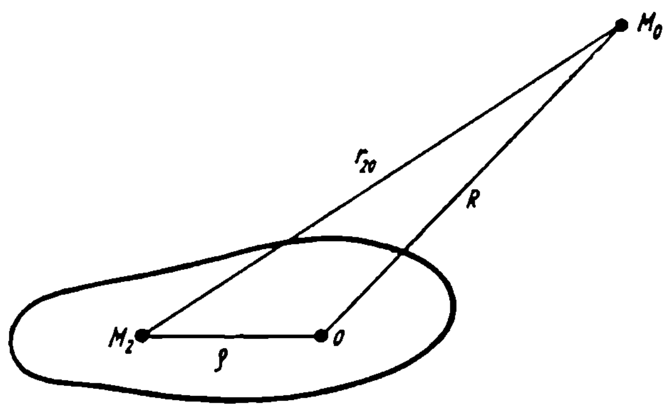

Fig. 22

We choose the center of mass in (D) as origin of a polar coordinate system (Fig. 22). If the polar axis $\Theta = 0$ passes through the point M_0, then

$$r_{20} = \sqrt{\varrho^2 + R^2 - 2R\varrho\cos\Theta}\,,$$

where R is the radius vector to the point M_0. We now expand $\frac{1}{r_{20}}$ in a power series in $\frac{1}{R}$:

$$\frac{1}{r_{20}} = \frac{1}{R\sqrt{1 - \frac{2\varrho}{R}\cos\Theta + \left(\frac{\varrho}{R}\right)^2}} = \frac{1}{R} + \frac{\varrho\cos\Theta}{R^2} + \cdots.$$

One then obtains:

$$\int\limits_{(D)} \frac{\mu\, d\tau}{r_{20}} = \frac{1}{R}\int\limits_{(D)} \mu\, d\tau + \frac{1}{R^2}\int\limits_{(D)} \mu\, \varrho \cos\Theta\, d\tau$$
$$+ \frac{1}{R^3}\cdot\frac{1}{2}\int\limits_{(D)} \mu \varrho^2(\cos 2\Theta + \cos^2\Theta)\, d\tau + \cdots$$

Since the origin of the coordinate system is the center of mass of (D), we have:

$$\int\limits_{(D)} \mu\, \varrho \cos\Theta\, d\tau = \int\limits_{(D)} \mu\, \zeta\, d\tau = 0\,.$$

We therefore obtain:

$$\int\limits_{(D)} \frac{\mu\, d\tau}{r_{20}} = \frac{M}{R} + \frac{1}{R^3}\cdot\frac{1}{2}\int\limits_{(D)} \mu \varrho^2(\cos 2\Theta + \cos^2\Theta)\, d\tau + \cdots$$

where M is the total mass in (D).

§13. On the First Derivatives of the Newtonian Potential

Theorem. *If the density μ of a Newtonian potential is bounded and integrable, then the potential possesses first derivatives with respect to x, y, and z; these derivatives are H-continuous in all of space.*

Proof. Let the point M_1 have coordinates $(x + h,y,z)$, and let r_{21} be its distance from the integration point $M_2(\xi,\eta,\zeta)$. We wish to investigate the difference

$$I = \frac{1}{h}\left[\int\limits_{(D)} \frac{\mu\, d\tau}{r_{21}} - \int\limits_{(D)} \frac{\mu\, d\tau}{r_{20}}\right] - \int\limits_{(D)} \mu\, \frac{\xi - x}{r_{20}^3}\, d\tau \tag{49}$$

and show that it becomes arbitrarily small in absolute value as $h \to 0$. If we have shown this, then it follows that

$$\lim_{h\to 0} \frac{1}{h}\left[\int\limits_{(D)} \frac{\mu\, d\tau}{r_{21}} - \int\limits_{(D)} \frac{\mu\, d\tau}{r_{20}}\right] = \int\limits_{(D)} \mu\, \frac{\xi - x}{r_{20}^3}\, d\tau\,, \tag{50}$$

wherewith the existence of the derivative with respect to x and its value is established.

If we construct the region (δ) of §11 about the point M_0, then

$$\int\limits_{(\delta)} \left|\mu\, \frac{\xi - x}{r_{20}^3}\right| d\tau < A \int\limits_{(\delta)} \frac{d\tau}{r_{20}^2} < 4\,\pi\, A\, \delta_2\,,$$

where A is an upper bound for $|\mu|$. This proves the convergence of the integral on the right-hand side of (50).

We now consider the sphere of radius $2\mid h\mid$ about the point M_0; we call this sphere $(2h)$. The difference I may now be written in the following form:

$$I = \frac{1}{h}\left[\int\limits_{(2h)} \mu \frac{d\tau}{r_{21}} - \int\limits_{(2h)} \mu \frac{d\tau}{r_{20}}\right] - \int\limits_{(2h)} \mu \frac{\xi - x}{r_{20}^3} d\tau + \int\limits_{(D-2h)} \mu \left\{\frac{1}{h}\left(\frac{1}{r_{21}} - \frac{1}{r_{20}}\right) - \frac{\xi - x}{r_{20}^3}\right\} d\tau . \tag{51}$$

The third integral in (51) is less than $4\pi A \cdot 2\mid h\mid$ in absolute value.

From the results of §11 it follows that

$$\left|\frac{1}{h}\int\limits_{(2h)} \mu \frac{d\tau}{r_{20}}\right| < \frac{1}{|h|} A \int\limits_{(2h)} \frac{d\tau}{r_{20}} = \frac{2\pi A}{|h|}(2|h|)^2 = 8\pi A |h| .$$

To estimate the first integral of (51), we consider a sphere of radius $3\mid h\mid$ about M_1. The first sphere $(2h)$ is contained in the interior of this sphere $(3h)$. From this it follows that

$$\left|\frac{1}{h}\int\limits_{(2h)} \mu \frac{d\tau}{r_{21}}\right| \leq \frac{1}{|h|}\int\limits_{(2h)} |\mu| \frac{d\tau}{r_{21}} \leq \frac{1}{|h|}\int\limits_{(3h)} |\mu| \frac{d\tau}{r_{21}} < \frac{A}{|h|}\, 2\pi (3|h|)^2 = 18\pi A |h| .$$

We now consider the last term in (51). With the help of TAYLOR's formula we find:

$$\frac{1}{h}\left(\frac{1}{r_{21}} - \frac{1}{r_{20}}\right) - \frac{\xi - x}{r_{20}^3} = \frac{h}{2}\left\{\frac{3(\xi - x - \Theta h)^2}{r'^5} - \frac{1}{r'^3}\right\},$$

where r' is the distance of a point M' between M_0 and M_1 with coordinates $(x + \Theta h, y, z)$ $(0 < \Theta < 1)$ from the point $M_2(\xi,\eta,\zeta)$. From this it follows that

$$\left|\frac{1}{h}\left(\frac{1}{r_{21}} - \frac{1}{r_{20}}\right) - \frac{\xi - x}{r_{20}^3}\right| < |h| \cdot \frac{4}{r'^3} .$$

The distance from M' to M_0 is less than $\mid h\mid$, while the point M_2 lies outside the sphere $(2h)$. On the basis of inequality (9) therefore

$$r' > \frac{1}{2} r_{20} ,$$

and hence

$$\left|\frac{1}{h}\left(\frac{1}{r_{21}} - \frac{1}{r_{20}}\right) - \frac{\xi - x}{r_{20}^3}\right| < |h| \cdot \frac{32}{r_{20}^3} .$$

If R is the radius of a sphere with center M_0 containing the region (D) in its interior, then

$$\left| \int\limits_{(D-2h)} \mu \left\{ \frac{1}{h} \left(\frac{1}{r_{21}} - \frac{1}{r_{20}} \right) - \frac{\xi - x}{r_{20}^3} \right\} d\tau \right|$$

$$< A \cdot 32 |h| \int\limits_0^{2\pi} \int\limits_0^{\pi} \int\limits_{2|h|}^{R} \frac{\varrho^2 \sin\Theta \, d\varrho \, d\Theta \, d\varphi}{\varrho^3}$$

$$= 128 \pi A |h| \ln \frac{R}{2|h|} > a A |h|^{\lambda'}.$$

It is now established that the difference I becomes arbitrarily small in absolute value as $|h| \to 0$, since the first three integrals in (51) go to zero as $cA \mid h \mid$ and the last as $cA \mid h \mid^{\lambda'}$. We have thus proved the existence of the derivative; its value is given by formula (50). In the same manner one proves the existence of the derivatives with respect to y and z.

It remains to establish the H-continuity of the derivatives. We prove the H-continuity of the derivative $\frac{\partial P}{\partial x}$. For this purpose, we denote the distance between M_0 and $M_1(x_1,y_1,z_1)$ by δ and consider the difference

$$I = \int\limits_{(D)} \mu \frac{\xi - x_1}{r_{21}^3} d\tau - \int\limits_{(D)} \mu \frac{\xi - x}{r_{20}^3} d\tau. \tag{52}$$

We consider the sphere (2δ) of radius 2δ about the point M_0 and write the difference (52) in the form

$$I = \int\limits_{(2\delta)} \mu \frac{\xi - x_1}{r_{21}^3} d\tau - \int\limits_{(2\delta)} \mu \frac{\xi - x}{r_{20}^3} d\tau + \int\limits_{(D-2\delta)} \mu \left\{ \frac{\xi - x_1}{r_{21}^3} - \frac{\xi - x}{r_{20}^3} \right\} d\tau. \tag{53}$$

Making use of the sphere (3δ) of radius 3δ about M_1, we find as in the discussion of the difference (51) that

$$\left| \int\limits_{(2\delta)} \mu \frac{\xi - x}{r_{20}^3} d\tau \right| < A \cdot 4\pi \cdot (2\delta),$$

$$\left| \int\limits_{(2\delta)} \mu \frac{\xi - x_1}{r_{21}^3} d\tau \right| \leq \int\limits_{(3\delta)} |\mu| \frac{|\xi - x_1|}{r_{21}^3} d\tau < A \cdot 4\pi \cdot (3\delta).$$

We now investigate the last integral in (53). Since $|\xi - x_1| \leqq r_{21}$, $|x - x_1| \leqq \delta$, and moreover $r_{21} > \frac{r_{20}}{2}$ in $(D - 2\delta)$,

$$\left|\frac{\xi - x_1}{r_{21}^3} - \frac{\xi - x}{r_{20}^3}\right|$$
$$= \left|\frac{1}{r_{20}^3}[(\xi - x_1) - (\xi - x)] + (\xi - x_1)\left(\frac{1}{r_{21}^3} - \frac{1}{r_{20}^3}\right)\right|$$
$$= \left|\frac{x - x_1}{r_{20}^3} + (\xi - x_1)\cdot\frac{r_{20} - r_{21}}{r_{20}\, r_{21}}\left(\frac{1}{r_{20}^2} + \frac{1}{r_{20}\, r_{21}} + \frac{1}{r_{21}^2}\right)\right|$$
$$< \frac{\delta}{r_{20}^3} + \frac{\delta}{r_{20}}\left(\frac{1}{r_{20}^2} + \frac{2}{r_{20}^2} + \frac{4}{r_{20}^2}\right) = \frac{8\,\delta}{r_{20}^3};$$

from this it follows that

$$\left|\int\limits_{(D-2\delta)}\mu\left\{\frac{\xi - x_1}{r_{21}^3} - \frac{\xi - x}{r_{20}^3}\right\}d\tau\right| < 8\,A\,\delta\int\limits_0^{2\pi}\int\limits_0^{\pi}\int\limits_{2\delta}^{R}\frac{\varrho^2\sin\Theta\, d\varrho\, d\Theta\, d\varphi}{\varrho^3}$$
$$= 32\,\pi A\,\delta\ln\frac{R}{2\delta} < b\,A\,\delta^{\lambda'}.$$

The difference (52) is therefore less in absolute value than a number of the form

$$a A\delta + bA\delta^{\lambda} \qquad (\lambda < 1),$$

and hence less than a number of the form $cA\delta^{\lambda}$, as was to be shown.

Supplement. One easily proves the following theorem:

Let the density μ be integrable (though possibly unbounded) in a certain finite region (D). If μ is bounded in a certain subregion (ω) of (D), then in every region (ω_1) which together with its boundary is contained in (ω) the potential P is bounded and has there bounded and continuous first derivatives.

Indeed,

$$P = \int\limits_{(D)}\frac{\mu\, d\tau}{r_{20}} = \int\limits_{(D-\omega)}\frac{\mu}{r_{20}}\,d\tau + \int\limits_{(\omega)}\frac{\mu}{r_{20}}\,d\tau.$$

As a Newtonian potential with bounded density, the second integral on the right-hand side has bounded and continuous derivatives everywhere in space.

The first integral on the right-hand side is bounded in the region (ω_1) and has continuous derivatives of every order there, since r_{21} for the point M_0 of (ω_1) has a positive lower bound if M_2 lies in $(D - \omega)$.

§14. On the Existence of the Second Derivatives of the Newtonian Potential

Simple continuity of the density alone is not sufficient for the existence of the second derivatives of the Newtonian potential. We demonstrate this with an example.

Let (D) be a sphere of radius $R < 1$ with center at the origin of the coordinate system. We put

$$\mu = \left(\frac{3\zeta^2}{\varrho^2} - 1\right) f(\varrho), \qquad \varrho = \sqrt{\xi^2 + \eta^2 + \zeta^2},$$

where $f(\varrho)$ is a continuous function on the interval $0 \leqq \varrho \leqq R$ with $f(0) = 0$. It is clear that μ is continuous inside the sphere and is equal to zero at the center. The first derivatives of the Newtonian potential with such a density were shown in the preceding section to be continuous everywhere. We shall show that one can choose the function $f(\varrho)$ in such a way that the second derivative of the Newtonian potential with respect to z does not exist at the center of the sphere.

For this purpose we compute the value $\psi(a)$ of the potential at the point $(0,0,a)$ with $0 < a < R$. Introducing polar coordinates, we have:

$$\psi(a) = 2\pi \int_0^R \varrho^2 f(\varrho) \left\{ \int_0^\pi \frac{[3\cos^2\Theta - 1]\sin\Theta}{\sqrt{a^2 + \varrho^2 - 2a\varrho\cos\Theta}}\, d\Theta \right\} d\varrho .$$

Putting $\sqrt{a^2 + \varrho^2 - 2a\varrho\cos\Theta} = t$, we obtain:

$$\int_0^\pi \frac{[3\cos^2\Theta - 1]\sin\Theta}{\sqrt{a^2 + \varrho^2 - 2a\varrho\cos\Theta}}\, d\Theta$$

$$= \frac{1}{a\varrho} \int_{|a-\varrho|}^{a+\varrho} \left[3\left(\frac{a^2 + \varrho^2 - t^2}{2a\varrho}\right)^2 - 1\right] dt = \begin{cases} \dfrac{4}{5}\dfrac{\varrho^2}{a^3}, & \text{if} \quad \varrho \leq a, \\[2ex] \dfrac{4}{5}\dfrac{a^2}{\varrho^3}, & \text{if} \quad \varrho \geq a. \end{cases}$$

From this it follows that

$$\psi(a) = \frac{8\pi}{5}\left[\frac{1}{a^3}\int_0^a \varrho^4 f(\varrho)\, d\varrho + a^2 \int_a^R \frac{f(\varrho)}{\varrho}\, d\varrho\right].$$

Differentiating this expression, we find after simplification:

$$\psi'(a) = \frac{8\pi}{5}\left[-\frac{3}{a^4}\int_0^a \varrho^4 f(\varrho)\, d\varrho + 2a \int_a^R \frac{f(\varrho)}{\varrho}\, d\varrho\right].$$

Because of the continuity of the first derivatives mentioned above, we obtain for any choice of the continuous function $f(\varrho)$:

$$\psi'(0) = \lim_{a\to 0} \psi'(a) = 0 .$$

We consider the quotient

$$\frac{\psi'(a) - \psi'(0)}{a} = \frac{8\pi}{5}\left[-\frac{3}{a^5}\int_0^a \varrho^4 f(\varrho)\,d\varrho + 2\int_a^R \frac{f(\varrho)}{\varrho}\,d\varrho\right].$$

The first summand in brackets has limit $-\frac{3}{5}f(0) = 0$ as $a \to 0$. The second summand has no limit if we choose $f(\varrho)$ such that the integral

$$\int_0^R \frac{f(\varrho)}{\varrho}\,d\varrho$$

diverges. It suffices to put $f(\varrho) = \frac{1}{\ln \varrho}$. With this choice of $f(\varrho)$ the derivative $\psi''(0)$ does not exist, and $\psi''(a)$ increases without bound as $a \to 0$. If we put

$$f(\varrho) = \frac{\cos\sqrt{\ln\frac{1}{\varrho}}}{\sqrt{\ln\frac{1}{\varrho}}}, \qquad f(0) = 0,$$

then $\psi''(0)$ again does not exist, but $\psi''(a)$ remains bounded.

Theorem. *If the density μ is H-continuous, i.e., if*

$$|\mu - \mu_0| < A r_{20}^{\lambda},$$

then the Newtonian potential possesses second derivatives at all points not lying on the boundary of the region (D).

Proof. It is obvious that the second derivatives exist at points outside the region (D). It remains to show that they also exist at points in the interior of (D). We shall study the derivative $\frac{\partial^2 P}{\partial x^2}$. Let $M_0(x,y,z)$ be an interior point of (D). Let (d) be a sphere about the point of radius d so small that the sphere is entirely contained in the interior of (D); we write the potential (46) in the form

$$P = \int_{(d)} \mu \frac{d\tau}{r_{20}} + \int_{(D-d)} \mu \frac{d\tau}{r_{20}} \cdot \tag{54}$$

It is clear that the second derivatives exist at the point M_0 in the second summand, since this is a harmonic function in (d). The derivative of this summand is equal to

$$\int_{(D-d)} \mu \left\{\frac{3(\xi - x)^2}{r_{20}^5} - \frac{1}{r_{20}^3}\right\} d\tau . \tag{55}$$

It remains to investigate the first summand in (54) which we shall denote by P_1. Let h be an arbitrary number satisfying the inequality $|h| < \frac{d}{4}$. The point with coordinates $(x + h, y, z)$ we denote by M_1 and its distance from M_2 by r_{21}.

We form the difference

$$\begin{aligned} \frac{1}{h}\left\{\int\limits_{(d)} \mu \frac{\xi - x_1}{r_{21}^3} d\tau - \int\limits_{(d)} \mu \frac{\xi - x}{r_{20}^3} d\tau\right\} &= \frac{\mu_0}{h}\left(\int\limits_{(d)} \frac{\xi - x_1}{r_{21}^3} d\tau - \int\limits_{(d)} \frac{\xi - x}{r_{20}^3} d\tau\right) \\ &+ \frac{1}{h}\left(\int\limits_{(d)} (\mu - \mu_0) \frac{\xi - x_1}{r_{21}^3} d\tau - \int\limits_{(d)} (\mu - \mu_0) \frac{\xi - x}{r_{20}^3} d\tau\right), \end{aligned} \tag{56}$$

where μ_0 is the value of μ at the point M_0. The limit of this difference as $h \to 0$ is precisely the second derivative $\frac{\partial^2 P_1}{\partial x^2}$ at the point M_0.

We first consider the first term on the right-hand side of (56). This is the product of $\frac{\mu_0}{h}$ and the difference of the first derivative of the Newtonian potential of the sphere (d) with unit density with respect to x' at the points M_1 and M_0, where the potential is considered as a function of the point $M'(x', y', z')$. If in the second line of formula (48) we replace a^2 by $(x' - x)^2 + (y' - y)^2 + (z' - z)^2$, differentiate with respect to x', then put $x' = x + h$ and $x' = x$ in succession, and form the difference, we obtain:

$$\frac{\mu_0}{h}\left(\int\limits_{(d)} \frac{\xi - x_1}{r_{21}^3} d\tau - \int\limits_{(d)} \frac{\xi - x}{r_{20}^3} d\tau\right) = -\frac{4\pi}{3}\mu_0. \tag{57}$$

We now consider the second summand in (56) and investigate the difference

$$\begin{aligned} &\frac{1}{h}\left(\int\limits_{(d)} (\mu - \mu_0) \frac{\xi - x_1}{r_{21}^3} d\tau - \int\limits_{(d)} (\mu - \mu_0) \frac{\xi - x}{r_{20}^3} d\tau\right) \\ &\quad - \int\limits_{(d)} (\mu - \mu_0)\left\{\frac{3(\xi - x)^2}{r_{20}^5} - \frac{1}{r_{20}^3}\right\} d\tau \\ &= \frac{1}{h}\int\limits_{(2h)} (\mu - \mu_0) \frac{\xi - x_1}{r_{21}^3} d\tau - \frac{1}{h}\int\limits_{(2h)} (\mu - \mu_0) \frac{\xi - x}{r_{20}^3} d\tau \\ &\quad - \int\limits_{(2h)} (\mu - \mu_0)\left\{\frac{3(\xi - x)^2}{r_{20}^5} - \frac{1}{r_{20}^3}\right\} d\tau \\ &\quad + \int\limits_{(d-2h)} (\mu - \mu_0)\left[\frac{1}{h}\left(\frac{\xi - x_1}{r_{21}^3} - \frac{\xi - x}{r_{20}^3}\right) - \left(\frac{3(\xi - x)^2}{r_{20}^5} - \frac{1}{r_{20}^3}\right)\right] d\tau; \end{aligned} \tag{58}$$

$(2h)$ here denotes the sphere of radius $2\,|\,h\,|$ about M_0 which on the basis of our choice of h is contained in (d).

We shall show that the difference (58) becomes arbitrarily small in absolute value as $h \to 0$, wherewith the limit of the second term of (56) will be determined.

For the third integral on the right-hand side of (58) we have:

$$\left| \int\limits_{(2h)} (\mu - \mu_0) \left\{ \frac{3(\xi - x)^2}{r_{20}^5} - \frac{1}{r_{20}^3} \right\} d\tau \right| < 4A \int\limits_0^{2\pi} \int\limits_0^{\pi} \int\limits_0^{2|h|} \frac{r_{20}^\lambda}{r_{20}^3} \varrho^2 \sin\Theta \, d\varrho \, d\Theta \, d\varphi$$

$$= 16\pi A \int\limits_0^{2|h|} \frac{d\varrho}{\varrho^{1-\lambda}} = \frac{16\pi}{\lambda} A (2\,|h|)^\lambda .$$

The sphere of radius $3\,|\,h\,|$ about M_1 is contained in (d) and contains the sphere $(2h)$ in its interior. From this and the inequality

$$|\mu - \mu_0| = |(\mu - \mu_1) + (\mu_1 - \mu_0)| \le |\mu - \mu_1| + |\mu_1 - \mu_0| < A r_{21}^\lambda + A\,|h|^\lambda$$

we obtain for the first integral:

$$\left| \frac{1}{h} \int\limits_{(2h)} (\mu - \mu_0) \frac{\xi - x_1}{r_{21}^3} d\tau \right| < \frac{A}{|h|} \int\limits_{(2h)} \frac{r_{21}^\lambda}{r_{21}^2} d\tau + \frac{A\,|h|^\lambda}{|h|} \int\limits_{(2h)} \frac{d\tau}{r_{21}^2}$$

$$< \frac{A}{|h|} \int\limits_{(3h)} \frac{r_{21}^\lambda}{r_{21}^2} d\tau + \frac{A\,|h|^\lambda}{|h|} \int\limits_{(3h)} \frac{d\tau}{r_{21}^2} = \frac{12\pi A}{1+\lambda} (3\,|h|)^\lambda + 12\pi A\,|h|^\lambda .$$

The absolute value of the second integral in (58) is less than

$$\frac{1}{|h|} \int\limits_{(2h)} \frac{|\mu - \mu_0|}{r_{20}^2} d\tau < \frac{4\pi A \cdot 2^{1+\lambda}}{1+\lambda} |h|^\lambda .$$

It now follows that the sum of the first three integrals on the right-hand side of (58) is less in absolute value than a number of the form $aA\,|\,h\,|^\lambda$. The function in the square brackets in the last integral remains bounded in the region of integration. Applying TAYLOR's formula, we find that

$$\frac{1}{h} \left(\frac{\xi - x_1}{r_{21}^3} - \frac{\xi - x}{r_{20}^3} \right) - \left(\frac{3(\xi - x)^2}{r_{20}^5} - \frac{1}{r_{20}^3} \right)$$

$$= \frac{h}{2} \left\{ \frac{15(\xi - x')^3}{r'^7} - \frac{9(\xi - x')}{r'^5} \right\},$$

where x', y, and z are the coordinates of a certain point M' in the interior of the segment M_0M_1 and r' is the distance of this point from M_2. Since now the distance $|\,M'M_0\,|$ is less than $|\,h\,|$ and the point M_2 lies outside the sphere

$(2h)$, it follows from inequality (9) that r' is greater than $\frac{1}{2}r_{20}$. We then have:

$$\left|\frac{1}{h}\left(\frac{\xi - x_1}{r_{21}^3} - \frac{\xi - x}{r_{20}^3}\right) - \left(\frac{3(\xi - x)^2}{r_{20}^5} - \frac{1}{r_{20}^3}\right)\right| < |h| \frac{24 \cdot 2^4}{r_{20}^4}.$$

The last integral in (58) is therefore less in absolute value than

$$24 \cdot 2^4 \cdot |h| \cdot A \int\limits_{(d-2h)} \frac{r_{20}^\lambda}{r_{20}^4}\, d\tau = A \cdot 24 \cdot 2^4 \cdot 4\pi |h| \int\limits_{2|h|}^{d} \frac{\varrho^2\, d\varrho}{\varrho^{4-\lambda}}$$

$$= \frac{24 \cdot 2^4 \cdot 4\pi}{1-\lambda} A \cdot |h| \left\{\frac{1}{(2|h|)^{1-\lambda}} - \frac{1}{d^{1-\lambda}}\right\} < cA|h|^\lambda.$$

We have herewith shown that the difference (58) actually becomes arbitrarily small in absolute values as $h \to 0$. We conclude from the results obtained that

$$\frac{\partial^2 P}{\partial x^2} = -\frac{4\pi}{3}\mu_0 + \int\limits_{(d)} (\mu - \mu_0) \left\{\frac{3(\xi - x)^2}{r_{20}^5} - \frac{1}{r_{20}^3}\right\} d\tau$$

$$+ \int\limits_{(D-d)} \mu \left\{\frac{3(\xi - x)^2}{r_{20}^5} - \frac{1}{r_{20}^3}\right\} d\tau.$$

We point out that in computing $\frac{\partial^2 P}{\partial x\, \partial y}$ the term corresponding to the first term on the right-hand side of (56) is equal to zero. We then find that

$$\frac{\partial^2 P}{\partial x \partial y} = 0 + \int\limits_{(d)} (\mu - \mu_0) \frac{3(\xi - x)(\eta - y)}{r_{20}^5} d\tau$$

$$+ \int\limits_{(D-d)} \mu \frac{3(\xi - x)(\eta - y)}{r_{20}^5} d\tau.$$

Remark. In place of the sphere (d) which contains the point M_0 one may choose any arbitrary region (D_0) which contains M_0 and is entirely contained in (D).

The formulas for $\frac{\partial^2 P}{\partial x^2}$ and $\frac{\partial^2 P}{\partial x\, \partial y}$ then assume the following form:

$$\frac{\partial^2 P}{\partial x^2} = \mu_0 \left(\frac{\partial^2}{\partial x^2} \int\limits_{(D_0)} \frac{d\tau}{r}\right)_{M_0} + \int\limits_{(D_0)} (\mu - \mu_0) \frac{\partial^2 \frac{1}{r_{20}}}{\partial x^2} d\tau + \int\limits_{(D-D_0)} \mu \frac{\partial^2 \frac{1}{r_{20}}}{\partial x^2} d\tau,$$

$$\frac{\partial^2 P}{\partial x \partial y} = \mu_0 \left(\frac{\partial^2}{\partial x \partial y} \int\limits_{(D_0)} \frac{d\tau}{r}\right)_{M_0} + \int\limits_{(D_0)} (\mu - \mu_0) \frac{\partial^2 \frac{1}{r_{20}}}{\partial x \partial y} d\tau + \int\limits_{(D-D_0)} \mu \frac{\partial^2 \frac{1}{r_{20}}}{\partial x \partial y} d\tau.$$

§15. The Theorem of Poisson

Theorem. (Poisson) *If the density μ of a Newtonian potential in a finite region (D) is H-continuous, then at every interior point M_0 of (D) the* Poisson *equation holds:*

$$(\Delta P)_{M_0} \equiv \left(\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2}\right)_{M_0} = -4\pi\mu_0 .$$

Proof. If we add the equations already obtained

$$\frac{\partial^2 P}{\partial x^2} = -\frac{4\pi}{3}\mu_0 + \int\limits_{(d)} (\mu - \mu_0)\frac{\partial^2 \frac{1}{r_{20}}}{\partial x^2}\, d\tau + \int\limits_{(D-d)} \mu \frac{\partial^2 \frac{1}{r_{20}}}{\partial x^2}\, d\tau ,$$

$$\frac{\partial^2 P}{\partial y^2} = -\frac{4\pi}{3}\mu_0 + \int\limits_{(d)} (\mu - \mu_0)\frac{\partial^2 \frac{1}{r_{20}}}{\partial y^2}\, d\tau + \int\limits_{(D-d)} \mu \frac{\partial^2 \frac{1}{r_{20}}}{\partial y^2}\, d\tau ,$$

$$\frac{\partial^2 P}{\partial z^2} = -\frac{4\pi}{3}\mu_0 + \int\limits_{(d)} (\mu - \mu_0)\frac{\partial^2 \frac{1}{r_{20}}}{\partial z^2}\, d\tau + \int\limits_{(D-d)} \mu \frac{\partial^2 \frac{1}{r_{20}}}{\partial z^2}\, d\tau ,$$

then we indeed arrive at the required equation.

Let $f(x,y,z)$ be continuous in a certain region (D) and let $M_0(x_0,y_0,z_0)$ be an interior point of this region. Let (M_0,h) be the sphere of radius h about M_0 where h is chosen so small that (M_0,h) is contained in (D). We wish to study the ratio of the difference of the mean value of the function $f(x,y,z)$ and the value of $f(x,y,z)$ at the point M_0 to the quantity $\frac{h^2}{10}$:

$$\Delta_h f = \left[\frac{1}{\frac{4}{3}\pi h^3}\int\limits_{(M_0,h)} f(M)\, d\tau - f(M_0)\right] : \frac{h^2}{10} = \frac{30}{4\pi h^5}\int\limits_{(M_0,h)} [f(M) - f(M_0)]\, d\tau .$$

If f is harmonic in (D), then it follows from the theorem on the mean value of harmonic functions that $\Delta_h f = 0$.

It is possible that $\Delta_h f$ for a continuous function $f(M)$ possesses a finite limit as $h \to 0$. One calls this limit the *generalized* Laplace *expression* of the function f. We shall denote this limit by $\Delta^* f(M_0)$.

If f has continuous derivatives of second order, then one sees easily that $\Delta^* f(M_0) = \Delta f(M_0)$, i.e., the generalized Laplace expression coincides with the usual one. We have in fact:

$$\begin{aligned} f(M) - f(M_0) = {} & f'_x(M_0)(x - x_0) + f'_y(M_0)(y - y_0) + f'_z(M_0)(z - z_0) \\ & + \tfrac{1}{2}[f''_{xx}(M_0)(x - x_0)^2 + f''_{yy}(M_0)(y - y_0)^2 + f''_{zz}(M_0)(z - z_0)^2 \\ & + 2f''_{xy}(M_0)(x - x_0)(y - y_0) + \cdots] \\ & + \tfrac{1}{2}\{[f''_{xx}(M') - f''_{xx}(M_0)](x - x_0)^2 + \cdots\}; \end{aligned}$$

Here M' is certain point in the interior of the segment M_0M. Since by hypothesis the second derivatives of the function f are continuous, the difference $f''_{xx}(M') - f''_{xx}(M_0)$ and those analogous to it possess limit zero as $h \to 0$. From this it follows that the quantity in braces is less in absolute value than $h^2\varepsilon(h)$, where $\lim_{h\to 0} \varepsilon(h) = 0$. Hence,

$$\left|\frac{15}{4\pi h^5}\int\limits_{(M_0,h)}\{\cdots\}\,d\tau\right| < 5\,\varepsilon(h) \to 0 \qquad (h \to 0).$$

One sees moreover that the integrals over (M_0,h) of

$$x - x_0, \quad y - y_0, \quad z - z_0, \quad (x - x_0)(y - y_0), \quad (x - x_0)(z - z_0),$$
$$(y - y_0)(z - z_0)$$

vanish, while for the integrals of $(x - x_0)^2$, $(y - y_0)^2$, and $(z - z_0)^2$, which are all the same, one obtains:

$$\frac{1}{3}\int\limits_{(M_0,h)}[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]\,d\tau = \frac{4\pi}{3}\int\limits_0^h r^4\,dr = \frac{4\pi h^5}{15}.$$

From all this it now follows that

$$\begin{aligned}\Delta_h f &= \frac{30}{4\pi h^5}\int\limits_{(M_0,h)}[f(M) - f(M_0)]\,d\tau \\ &= \frac{30}{4\pi h^5}\left[\frac{1}{2}\cdot\frac{4\pi h^5}{15}\cdot(f''_{xx}(M_0) + f''_{yy}(M_0) + f''_{zz}(M_0))\right] \\ &+ \frac{15}{4\pi h^5}\int\limits_{(M_0,h)}\{\cdots\}\,d\tau \to f''_{xx}(M_0) + f''_{yy}(M_0) + f''_{zz}(M_0) = \Delta f \qquad (h \to 0).\end{aligned}$$

Hence, $\Delta^* f = \Delta f$, as was to be shown.

Theorem. *If the density μ of a Newtonian potential is bounded, integrable, and continuous at an interior point M_0 of the region (D), then*

$$\Delta^* P(M_0) = -4\pi\mu_0,$$

i.e., at every point of continuity of the density the Newtonian *potential satisfies* Poisson's *equation with the generalized* Laplace *expression.*

Proof. Denoting by μ_0 the value of μ at the point M_0, we have:

$$P(M_1) = \int\limits_{(D)}\frac{\mu(2)}{r_{12}}\,d\tau_2 = \mu_0\int\limits_{(D)}\frac{d\tau_2}{r_{12}} + \int\limits_{(D)}(\mu - \mu_0)\,\frac{d\tau_2}{r_{12}}.$$

The integral

$$\int\limits_{(D)}\frac{d\tau_2}{r_{12}},$$

as a Newtonian potential with density 1, has continuous second derivatives; its LAPLACE expression is equal to -4π. Its generalized LAPLACE expression therefore also has the value -4π, so that to prove the theorem it suffices to show that the generalized LAPLACE expression of the second integral at the point M_0 is equal to zero. It is to be shown therefore that

$$\frac{1}{h^5}\int\limits_{(M_0,\,h)}\left[\int\limits_{(D)}(\mu-\mu_0)\left(\frac{1}{r_{12}}-\frac{1}{r_{02}}\right)d\tau_2\right]d\tau_1 \to 0 \qquad (h\to 0).$$

We note first of all that it is here permissible to interchange the order of integration, since the inner integral is bounded when the integrand is replaced by its absolute value. Moreover,

$$\int\limits_{(M_0,h)}\frac{1}{r_{12}}\,d\tau_1=\begin{cases}\dfrac{4\pi h^3}{3}\cdot\dfrac{1}{r_{02}}, & \text{for } M_2 \text{ outside } (M_0,h),\\[2ex] 2\pi h^2-\dfrac{2\pi}{3}r_{02}^2, & \text{for } M_2 \text{ inside } (M_0,h).\end{cases}$$

We obtain therefore:

$$\begin{aligned}
&\frac{1}{h^5}\int\limits_{(M_0,h)}\left[\int\limits_{(D)}(\mu-\mu_0)\left(\frac{1}{r_{12}}-\frac{1}{r_{02}}\right)d\tau_2\right]d\tau_1\\
&\quad=\frac{1}{h^5}\int\limits_{(D)}(\mu-\mu_0)\left[\int\limits_{(M_0,h)}\left(\frac{1}{r_{12}}-\frac{1}{r_{02}}\right)d\tau_1\right]d\tau_2\\
&\quad=\frac{1}{h^5}\int\limits_{(D-(M_0,h))}(\mu-\mu_0)\left[\frac{4\pi h^3}{3}\cdot\frac{1}{r_{02}}-\frac{4\pi h^3}{3}\cdot\frac{1}{r_{02}}\right]d\tau_2\\
&\quad\quad+\frac{1}{h^5}\int\limits_{(M_0,h)}(\mu-\mu_0)\left[2\pi h^2-\frac{2\pi}{3}r_{02}^2-\frac{4\pi h^3}{3}\cdot\frac{1}{r_{02}}\right]d\tau_2\\
&\quad=\frac{1}{h^5}\int\limits_{(M_0,h)}(\mu-\mu_0)\left[2\pi h^2-\frac{2\pi}{3}r_{02}^2-\frac{4\pi h^3}{3}\cdot\frac{1}{r_{02}}\right]d\tau_2.
\end{aligned}$$

Since μ is continuous at the point M_0, for every $\varepsilon>0$ there exists a sufficiently small $h>0$ such that $|\mu-\mu_0|<\varepsilon$ in (M_0,h). Hence, the absolute value of the last integral is less than

$$\frac{\varepsilon}{h^5}\left[2\pi h^2\cdot\frac{4}{3}\pi h^3+\frac{2\pi}{3}\cdot\frac{4\pi h^5}{5}+\frac{4\pi h^3}{3}\cdot 2\pi h^2\right]=\frac{88}{15}\pi^2\varepsilon.$$

The integral therefore has limit zero as $h\to 0$, wherewith the theorem is proved.

The generalized LAPLACE expression was introduced by I. I. PRIVALOV[1] to whom the theorem is due.

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[1] I. I. PRIVALOV, *Subgarmonicheskie funktsii* (Subharmonic Functions), Ch. I, §2, Moscow, 1937.

Subsequently a definition of generalized derivatives in the sense of Academician S. L. SOBOLEV will be given, and it will be shown that a Newtonian potential with square-summable density also satisfies the POISSON equation in a certain sense.

§16. On the Continuity of the Second Derivatives of the Newtonian Potential

Up until now in studying the Newtonian potential the boundary of the region (D) could be arbitrary. In the following theorem it is assumed that the region (D) in which the density μ is defined is bounded by a finite number of closed LYAPUNOV surfaces.

Theorem. *If the density μ of a Newtonian potential is H-continuous, then the second derivatives of the potential are H-continuous in the interior of (D_i) and in the interior of (D_e).*

The proof of the theorem is rather complicated; it can be found in Appendix III.

If we assume that the theorem is true, then we may conclude that the second derivatives have definite limits as the point M_0 approaches the boundary. These limits are different for points M_0 which approach the boundary from the interior of (D_i) and for those which approach the boundary from the interior of (D_e). The theorem of HUGONIOT-HADAMARD makes it possible to determine the jumps of these derivatives when the point M_0 passes through the boundary. The derivatives

$$\frac{\partial P}{\partial x}, \quad \frac{\partial P}{\partial y}, \quad \frac{\partial P}{\partial z}$$

can be interpreted as the values of two functions of which one is defined in the interior of (D_i), the other is defined in the interior of (D_e), and the two agree on the boundary.

If we denote by λ, ν, and ϑ certain functions defined on the boundary, then we may write:

$$\left.\begin{aligned}
&\frac{\partial^2 P_i}{\partial x^2} - \frac{\partial^2 P_e}{\partial x^2} = \lambda \cos(Nx), \qquad \frac{\partial^2 P_i}{\partial x \partial y} - \frac{\partial^2 P_e}{\partial x \partial y} = \nu \cos(Nx),\\
&\frac{\partial^2 P_i}{\partial x \partial y} - \frac{\partial^2 P_e}{\partial x \partial y} = \lambda \cos(Ny), \qquad \frac{\partial^2 P_i}{\partial y^2} - \frac{\partial^2 P_e}{\partial y^2} = \nu \cos(Ny),\\
&\frac{\partial^2 P_i}{\partial x \partial z} - \frac{\partial^2 P_e}{\partial x \partial z} = \lambda \cos(Nz), \qquad \frac{\partial^2 P_i}{\partial y \partial z} - \frac{\partial^2 P_e}{\partial y \partial z} = \nu \cos(Nz),\\
&\qquad\qquad \frac{\partial^2 P_i}{\partial x \partial z} - \frac{\partial^2 P_e}{\partial x \partial z} = \vartheta \cos(Nx),\\
&\qquad\qquad \frac{\partial^2 P_i}{\partial y \partial z} - \frac{\partial^2 P_e}{\partial y \partial z} = \vartheta \cos(Ny),\\
&\qquad\qquad \frac{\partial^2 P_i}{\partial z^2} - \frac{\partial^2 P_e}{\partial z^2} = \vartheta \cos(Nz).
\end{aligned}\right\} \tag{59}$$

If one equates the right-hand sides of those equations the left-hand sides of which are equal, one finds:

$$\begin{aligned}
\lambda \cos(Ny) &= \nu \cos(Nx),\\
\lambda \cos(Nz) &= \vartheta \cos(Nx),\\
\nu \cos(Nz) &= \vartheta \cos(Ny),
\end{aligned}$$

i.e.,

$$\frac{\lambda}{\cos(Nx)} = \frac{\nu}{\cos(Ny)} = \frac{\vartheta}{\cos(Nz)} = \Theta.$$

From this it follows that

$$\lambda = \Theta \cos(Nx), \quad \nu = \Theta \cos(Ny), \quad \vartheta = \Theta \cos(Nz).$$

Adding the first, fifth, and final equations of (59), we find:

$$\left(\frac{\partial^2 P_i}{\partial x^2} + \frac{\partial^2 P_i}{\partial y^2} + \frac{\partial^2 P_i}{\partial z^2}\right) - \left(\frac{\partial^2 P_e}{\partial x^2} + \frac{\partial^2 P_e}{\partial y^2} + \frac{\partial^2 P_e}{\partial z^2}\right)$$
$$= \Theta[\cos^2(Nx) + \cos^2(Ny) + \cos^2(Nz)] = \Theta.$$

Applying the theorem of POISSON and recalling the fact that the potential is a harmonic function outside (D_i), it follows that

$$\Theta = -4\pi\mu.$$

From the foregoing we conclude:

$$\begin{aligned}
\frac{\partial^2 P_i}{\partial x^2} - \frac{\partial^2 P_e}{\partial x^2} &= -4\pi\mu \cos^2(Nx),\\
\frac{\partial^2 P_i}{\partial x \partial y} - \frac{\partial^2 P_e}{\partial x \partial y} &= -4\pi\mu \cos(Nx)\cos(Ny),\\
\frac{\partial^2 P_i}{\partial x \partial z} - \frac{\partial^2 P_e}{\partial x \partial z} &= -4\pi\mu \cos(Nx)\cos(Nz),\\
\frac{\partial^2 P_i}{\partial y^2} - \frac{\partial^2 P_e}{\partial y^2} &= -4\pi\mu \cos^2(Ny),\\
\frac{\partial^2 P_i}{\partial y \partial z} - \frac{\partial^2 P_e}{\partial y \partial z} &= -4\pi\mu \cos(Ny)\cos(Nz),\\
\frac{\partial^2 P_i}{\partial z^2} - \frac{\partial^2 P_e}{\partial z^2} &= -4\pi\mu \cos^2(Nz).
\end{aligned}$$

§17. The Derivatives of the Newtonian Potential with Differentiable Density

We assume that the density μ is bounded and has continuous first derivatives. If we consider a sphere (δ) of radius δ about the point M_0 we may write:

$$\frac{\partial P}{\partial x} = \int\limits_{(D-\delta)} \mu \frac{\xi - x}{r_{20}^3} d\tau + \int\limits_{(\delta)} \mu \frac{\xi - x}{r_{20}^3} d\tau .$$

The second integral is less in absolute value than $4\pi A\delta$. Rearranging the first integral, we obtain:

$$\begin{aligned} \int\limits_{(D-\delta)} \mu \frac{\xi - x}{r_{20}^3} d\tau &= \int\limits_{(D-\delta)} \mu \frac{\partial \frac{1}{r_{20}}}{\partial x} d\tau = -\int\limits_{(D-\delta)} \mu \frac{\partial \frac{1}{r_{20}}}{\partial \xi} d\tau \\ &= -\int\limits_{(D-\delta)} \frac{\partial \left(\mu \frac{1}{r_{20}}\right)}{\partial \xi} d\tau + \int\limits_{(D-\delta)} \frac{\partial \mu}{\partial \xi} \frac{d\tau}{r_{20}} . \end{aligned} \tag{60}$$

We first consider the second integral which can be written in the form

$$\int\limits_{(D)} \frac{\partial \mu}{\partial \xi} \frac{d\tau}{r_{20}} - \int\limits_{(\delta)} \frac{\partial \mu}{\partial \xi} \frac{d\tau}{r_{20}} .$$

The second term is less in absolute value than $2\pi A\delta^2$, where A is an upper bound for $|\mu|$ and $\left|\frac{\partial \mu}{\partial \xi}\right|$.

For the first integral in (60) we have:

$$-\int\limits_{(D-\delta)} \frac{\partial \left(\mu \frac{1}{r_{20}}\right)}{\partial \xi} d\tau = -\int\limits_{(S)} \mu \cos(Nx) \frac{d\sigma}{r_{20}} + \int\limits_{(\sigma)} \mu \cos(Nx) \frac{d\sigma}{r_{20}} ;$$

here (σ) is the surface of the sphere (δ), and N is the outward normal of (σ).

The absolute value of the last integral is clearly less than

$$A \int\limits_{(\sigma)} \frac{d\sigma}{r_{20}} = \frac{A}{\delta} \int\limits_0^{2\pi} \int\limits_0^{\pi} \delta^2 \sin\Theta \, d\Theta \, d\varphi = 4\pi A\delta .$$

As δ tends to zero, we therefore obtain:

$$\frac{\partial P}{\partial x} = -\int\limits_{(S)} \mu \cos(Nx) \frac{d\sigma}{r_{20}} + \int\limits_{(D)} \frac{\partial \mu}{\partial \xi} \frac{d\tau}{r_{20}} . \tag{61}$$

We now assume that μ possesses bounded and continuous second derivatives; differentiating and again applying formula (61), we find:

$$\frac{\partial^2 P}{\partial x^2} = -\int\limits_{(S)} \mu \cos(Nx) \frac{\partial \frac{1}{r_{20}}}{\partial x} d\sigma - \int\limits_{(S)} \frac{\partial \mu}{\partial \xi} \cos(Nx) \frac{d\sigma}{r_{20}} + \int\limits_{(D)} \frac{\partial^2 \mu}{\partial \xi^2} \frac{d\tau}{r_{20}}.$$

In the same manner one finds:

$$\frac{\partial^2 P}{\partial y^2} = -\int\limits_{(S)} \mu \cos(Ny) \frac{\partial \frac{1}{r_{20}}}{\partial y} d\sigma - \int\limits_{(S)} \frac{\partial \mu}{\partial \eta} \cos(Ny) \frac{d\sigma}{r_{20}} + \int\limits_{(D)} \frac{\partial^2 \mu}{\partial \eta^2} \frac{d\tau}{r_{20}},$$

$$\frac{\partial^2 P}{\partial z^2} = -\int\limits_{(S)} \mu \cos(Nz) \frac{\partial \frac{1}{r_{20}}}{\partial z} d\sigma - \int\limits_{(S)} \frac{\partial \mu}{\partial \zeta} \cos(Nz) \frac{d\sigma}{r_{20}} + \int\limits_{(D)} \frac{\partial^2 \mu}{d \zeta^2} \frac{d\tau}{r_{20}}.$$

Termwise addition of these equations now gives:

$$\Delta P = -\int\limits_{(S)} \mu \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma - \int\limits_{(S)} \frac{d\mu_i}{dn} \frac{d\sigma}{r_{20}} + \int\limits_{(D)} \Delta\mu \frac{d\tau}{r_{20}}.$$

We obtained this last equation under the hypothesis that the second derivatives of μ were continuous. This restriction was necessary inasmuch as we have not generalized GREEN's formula to the case of functions possessing only bounded and integrable rather than continuous derivatives. If such a generalization had been made, then one could conclude that the last formula is valid as long as the second derivatives of the function μ are bounded and integrable.

If the point M_0 lies in the interior of (D_i) we find from the last equation:

$$\mu = -\frac{1}{4\pi} \int\limits_{(D)} \Delta\mu \frac{d\tau}{r_{20}} + \frac{1}{4\pi} \int\limits_{(S)} \mu \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma + \frac{1}{4\pi} \int\limits_{(S)} \frac{d\mu_i}{dn} \frac{d\sigma}{r_{20}}. \tag{62}$$

If the point M_0 lies outside (D_i), then

$$0 = -\frac{1}{4\pi} \int\limits_{(D)} \Delta\mu \frac{d\tau}{r_{20}} + \frac{1}{4\pi} \int\limits_{(S)} \mu \frac{\cos(r_{20} N)}{r_{20}^2} d\sigma + \frac{1}{4\pi} \int\limits_{(S)} \frac{d\mu_i}{dn} \frac{d\sigma}{r_{20}}. \tag{63}$$

Formula (62) makes it possible to represent every function which is defined in (D_i) and has second derivatives there in the form of a sum of three potentials.

Finally, we shall assume that the function μ satisfies the following three conditions:

1. The first derivatives of μ are continuous everywhere.
2. At every point not on the boundary (S) μ has bounded and continuous second derivatives.

3. Outside (D) μ is a harmonic function.
From condition 1 it follows that on the boundary (S):

$$\frac{d\mu_e}{dn} = \frac{d\mu_i}{dn}.$$

If the point M_0 lies outside (D), then since μ is harmonic there we find:

$$\mu = -\frac{1}{4\pi}\int_{(S)} \mu \frac{\cos(r_{20}N)}{r_{20}^2}\, d\sigma - \frac{1}{4\pi}\int_{(S)} \frac{d\mu_i}{dn}\frac{d\sigma}{r_{20}}. \tag{63'}$$

If the point M_0 lies in the interior of (D), then

$$0 = -\frac{1}{4\pi}\int_{(S)} \mu \frac{\cos(r_{20}N)}{r_{20}^2}\, d\sigma - \frac{1}{4\pi}\int_{(S)} \frac{d\mu_i}{dn}\frac{d\sigma}{r_{20}}. \tag{62'}$$

Adding formulas (63′) and (63) and (62′) and (62), one finds that the following relation is valid everywhere:

$$\mu = -\frac{1}{4\pi}\int_{(D)} \Delta\mu \frac{d\tau}{r_{20}}.$$

§18. The Function Classes $H(l,A,\lambda)$ and the Surfaces L_k

To simplify notation and the formulation of subsequent theorems we introduce the following definitions.

Let a function $f(x,y,z) = f(M)$ defined in a region (D) be bounded and possess bounded and continuous derivatives up to order l $(l \geqq 0)$, where the derivatives of order l are H-continuous; i.e.,

$$\left|\frac{\partial^p f}{\partial x^{p_1}\partial y^{p_2}\partial z^{p_3}}\right| < A \qquad \begin{pmatrix} p_1 + p_2 + p_3 = p \\ p = 0, 1, 2, \ldots, l \end{pmatrix}, \tag{64}$$

and for any pair of points M_1 and M_2 of (D) a distance r_{12} apart less than a certain number $r_0 \leqq 1$ the inequality

$$\left|\left(\frac{\partial^l f}{\partial x^{l_1}\partial y^{l_2}\partial z^{l_3}}\right)_{M_1} - \left(\frac{\partial^l f}{\partial x^{l_1}\partial y^{l_2}\partial z^{l_3}}\right)_{M_2}\right| < A r_{12}^{\lambda} \quad (0 < \lambda \leq 1) \tag{65}$$

holds, where the numbers A and λ are independent of the choice of the points. We shall say that f *belongs to the class* $H(l,A,\lambda)$ *in* (D) and write $f \in H(l,A,\lambda)$.

Since under orthogonal coordinate transformation a derivative of any order of f in one coordinate system is a linear combination with bounded coefficients of derivatives of f of the same order in the other coordinate system, we come easily to the following conclusion: If inequalities (64) and (65) hold in one system, then analogous inequalities hold in any other Cartesian coordinate system; A is then simply to be replaced by cA where $c > 1$ and depends only on l. Since this factor c is unimportant for the following considerations, we

shall assume that inequalities (64) and (65) hold in any coordinate system, as soon as we have shown that they hold in a particular system.

The class $H(l,A,\lambda)$ is similarly defined for functions of two variables.

Now let (S) be a closed LYAPUNOV surface, and let N be its outward normal. We choose an arbitrary point M_0 of (S) as origin of a rectangular coordinate system (ξ,η,ζ) with the ζ axis along the normal N_0 at the point M_0 and the ξ and η axes fixed in any way in the tangent plane. It then follows from I, §1 that the subregion (Σ) of (S) contained in a LYAPUNOV sphere about M_0 is cut at most once by any line parallel to the ζ axis and that the projection of this subregion on the (ξ,η) plane fills a region which contains the circle (Λ_0) about M_0 with radius $d_0 = \frac{7}{9}d$. We shall denote the subregion of (Σ) with projection the circle (Λ_0) by (Σ_0), and we shall subsequently assume for simplicity that $d_0 \leqq \frac{1}{2}$; the distance between any two points of (Λ_0) is then not greater than 1.

Let a function μ be given on (S). If ξ, η, and ζ are coordinates of a point M of (Σ_0) we define μ in (Λ_0) as a function of ξ and η by putting $\mu(\xi,\eta) = \mu(M)$.

We shall say that *a function μ defined on (S) belongs to the class $H(l,A,\lambda)$ on (S) if $\mu(\xi,\eta) \in H(l,A,\lambda)$ in (Λ_0) and A and λ are independent of the choice of the point M_0*.

Let M_1 and M_2 be two points of (Σ_0) a distance r apart. Let ϱ be the projection of the segment M_1M_2 onto the (ξ,η) plane; we then have that

$$\varrho \leq r < \frac{\varrho}{\sin\omega} < 2\,\varrho$$

and hence $\varrho^\lambda \leqq r^\lambda < 2^\lambda \varrho^\lambda$. From this it follows that an H-continuous function on (S) belongs to the class $H(0,A,\lambda)$, and conversely that a function belonging to the class $H(0,A,\lambda)$ on (S) is H-continuous.

If the function $f(x,y,z) \in H(0,A,\lambda)$ in (D_i), i.e., if it is H-continuous in (D_i), then its boundary value (f) on the boundary (S)—as was shown in I, §2—is an H-continuous function on (S); hence, on (S) $(f) \in H(0,cA,\lambda)$.

It is clear that functions of the class $H(l,A,\lambda)$ possess the following properties:

1. If $\mu_1 \in H(l,A_1,\lambda)$ and $\mu_2 \in H(l,A_2,\lambda)$, then

$\mu_1 \in H(l,A_1,\lambda)$, $\mu_2 \in H(l,A_2,\lambda)$, hence

$$\text{a)}\quad \mu_1 + \mu_2 \in H(l,A_1+A_2,\lambda),$$
$$\text{b)}\quad \mu_1 \cdot \mu_2 \in H(l,cA_1A_2,\lambda),$$

where $c > 1$ and depends only on l.

2. If $\mu \in H(l,A,\lambda)$, then $\mu \in H(l,A,\lambda')$ where λ' is an arbitrary number in the interval $0 < \lambda' < \lambda$.

3. If $\mu \in H(l,A,\lambda)$ on (S), then $\mu \in H(l',A\sqrt{2},1)$ on (S) where l' is an arbitrary number in the interval $0 \leqq l' < l$. If the region (D) is convex, then a function f which belongs to the class $H(l,A,\lambda)$ in (D) possesses an analogous property;

in this case $A\sqrt{2}$ has only to be replaced by $A\sqrt{3}$. If (D) is bounded by a LYAPUNOV surface, then, as follows from I, §2,

$$f \in H\left(l',\ \frac{2\sqrt{3}\,A}{\sin\omega},\ 1\right).$$

We now come to another definition. The equation for (Σ_0) in the (ξ,η,ζ) system has the form

$$\zeta = F(\xi,\eta). \tag{66}$$

We shall say that (S) *belongs to the class* $L_k(B,\lambda)$ *if* $F(\xi,\eta) \in H(k,B,\lambda)$ *and the numbers* B *and* λ *are independent of the choice of the point* M_0.

The LYAPUNOV surfaces are surfaces of the class $L_1(B,\lambda)$.

Let now (x,y,z) be a fixed rectangular Cartesian coordinate system.

Lemma 1. *If* $(S) \in L_k(B,\lambda)$, *then* $\cos(Nx) \in H(k-1,c,\lambda)$ *where the number* c *depends only on* B *and* k.

Proof. Let (ξ,η,ζ) be a local coordinate system about a certain point M_0 of (S). Since

$$\cos(Nx) = \cos(N\xi)\cos(\xi x) + \cos(N\eta)\cos(\eta x) + \cos(N\zeta)\cos(\zeta x)$$

and $\cos(\xi x)$, $\cos(\eta x)$, and $\cos(\zeta x)$ are constant for a fixed (ξ,η,ζ) system and are not greater than one in absolute value, it suffices to show that in (Λ_0)

$$\cos(N\xi) \in H(k-1, c_1, \lambda), \quad \cos(N\eta) \in H(k-1, c_2, \lambda),$$
$$\cos(N\zeta) \in H(k-1, c_3, \lambda),$$

where c_1, c_2, and c_3 depend only on B and k. Now

$$\cos(N\xi) = \frac{-F'_\xi}{\sqrt{1+(F'_\xi)^2+(F'_\eta)^2}},$$

whence it follows that $\cos(N\xi)$ as a function of ξ and η possesses continuous derivatives up to order $k-1$ in (Λ_0).

Let q be the number of different derivatives of $F(\xi,\eta)$ up to order k. With the notation

$$\alpha_1 = F'_\xi, \qquad \alpha_2 = F'_\eta, \qquad \alpha_3 = F''_{\xi^2}, \ldots, \alpha_q = F^{(k)}_{\eta^k}$$

we then have:

$$\frac{\partial^l \cos(N\xi)}{\partial\xi^{l_1}\partial\eta^{l_2}} = \frac{R_{l_1 l_2}(\alpha_1,\ldots,\alpha_q)}{(\sqrt{1+\alpha_1^2+\alpha_2^2})^{2l+1}} = \Phi_{l_1 l_2}(\alpha_1,\ldots,\alpha_q),$$

where $R_{l_1 l_2}$ is a polynomial function of its arguments. Hence $\Phi_{l_1 l_2}(\alpha_1,\ldots,\alpha_q)$ is bounded and has bounded first derivatives in the region $|\alpha_i| \leqq B$. Let c_1' be the maximum of $|\Phi_{l_1 l_2}(\alpha_1,\ldots,\alpha_q)|$ for $|\alpha_i| \leqq B$ and $l_1 + l_2 \leqq k-1$ and c_1'' be the maximum of $\left|\frac{\partial\Phi_{k_1 k_2}}{\partial\alpha_i}\right|$ for $|\alpha_i| \leqq B$ and $k_1 + k_2 = k-1$. If we denote

the values of α_i at the points (ξ_1,η_1) and (ξ_2,η_2) of (Λ_0) by α_i' and α_i'' respectively, then

$$|\alpha_i' - \alpha_i''| < B\sqrt{2}\,\varrho_{12}^\lambda \qquad \text{with} \qquad \varrho_{12} = \sqrt{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2}.$$

Therefore,

$$\left| \left(\frac{\partial^{k-1}\cos(N\xi)}{\partial\xi^{k_1}\partial\eta^{k_2}}\right)_{(\xi_1,\eta_1)} - \left(\frac{\partial^{k-1}\cos(N\xi)}{\xi^{k_1}\partial\eta^{k_2}}\right)_{(\xi_2,\eta_2)} \right|$$
$$= \left| \sum_{i=1}^{q} \left(\frac{\partial \Phi_{k_1k_2}}{\partial\alpha_i}\right)' (\alpha_i' - \alpha_i'') \right| < q\sqrt{2}\,c_1''\,B\,\varrho_{12}^\lambda .$$

If we now introduce the symbol c_1 for the larger of the two numbers c_1' and $q\sqrt{2}c_1''B$, then $\cos(N\xi) \in H(k-1,c_1,\lambda)$. We prove a similar thing for $\cos(N\eta)$ and $\cos(N\zeta)$. Then

$$\cos(Nx) \in H(k-1, c, \lambda) \text{ with } c = \sqrt{c_1^2 + c_2^2 + c_3^2}.$$

Similarly, $\cos(Ny) \in H(k-1,c,\lambda)$, $\cos(Nz) \in H(k-1,c,\lambda)$.

Lemma 2. *If $\mu \in H(l,A,\lambda')$ on (S) and $(S) \in L_k(B,\lambda)$ with $\lambda' \leqq \lambda$ and $l \leqq k$, then*

$$D_x\mu \in H(l-1, aA, \lambda'), \tag{67}$$

where a depends only on B and k.

Proof. Again let (ξ,η,ζ) be a local coordinate system about the point M_0 on (S). Then for μ on (Σ_0), $\mu = \mu(\xi,\eta)$.

If we consider $\mu(\xi,\eta)$ a function which does not depend on ζ but is defined in the cylinder with (Λ_0) as base and the ζ axis as axis of the cylinder, then

$$\begin{aligned} D_x\mu = \left(\frac{\partial\mu}{\partial x}\right) - \frac{d\mu}{dn}\cos(Nx) = \left[\frac{\partial\mu}{\partial\xi}\cos(\xi,x) + \frac{\partial\mu}{\partial\eta}\cos(\eta,x)\right] \\ - \left[\frac{\partial\mu}{\partial\xi}\cos(N\xi) + \frac{\partial\mu}{\partial\eta}\cos(N\eta)\right]\cos(Nx). \end{aligned} \tag{68}$$

It is clear that in (Λ_0) $\dfrac{\partial\mu}{\partial\xi}$ and $\dfrac{\partial\mu}{\partial\eta} \in H(l-1,A\lambda')$. Moreover, in (Λ_0)

$$\cos(N\xi),\ \cos(N\eta),\ \cos(Nx) \in H(k-1,c,\lambda),$$

and hence in (Λ_0)

$$\cos(N\xi),\ \cos(N\eta),\ \cos(Nx) \in H(l-1, c\sqrt{2}, \lambda')$$

where c depends only on B and k.

Making use of properties 1a and 1b for sums and products of functions of the class $H(l,A,\lambda)$ and the fact that $\cos(\xi x)$ and $\cos(\eta x)$ are constant and no greater than one in absolute value, then from (68) we find that

$$D_x\mu \in H(l-1, 2A + 2CA(c\sqrt{2})^2, \lambda').$$

Hence, (67) holds with $a = 2 + 4c^2C$.

We shall denote the boundary values on (S) of a function defined in (D_i) or in (D_e) by (f).

Lemma 3. *If $(S) \in L_k(B,\lambda)$ and if $f(x,y,z) \in H(l,A,\lambda)$ in (D_i) with $l \leqq k$, then, $(f) \in H(l,cA,\lambda)$ on (S) where c depends only on B and l.*

Proof. Let (ξ,η,ζ) be a local coordinate system about a point M_0 of (S). For sufficiently small positive ϵ the surface

$$\zeta = F(\xi, \eta) - \varepsilon$$

lies in the interior of (D_i). Introducing the notation $f_\epsilon(\xi,\eta) = f(\xi,\eta,F(\xi,\eta) - \varepsilon)$, we obtain for (f) on (Σ_0):

$$(f) = \lim_{\varepsilon \to 0} f_\varepsilon(\xi, \eta).$$

In (Λ_0) the function $f_\epsilon(\xi,\eta)$ has continuous derivatives of order l with respect to ξ and η, while f_ϵ and its derivatives up to order l tend uniformly to (f) and the corresponding derivatives of (f) with respect to ξ and η respectively as $\epsilon \to 0$; (f) is hereby to be considered as a function in (Λ_0). Each derivative of (f) is hence the sum of a finite number of summands of the form

$$\varphi(\xi, \eta) = \left(\frac{\partial^p f}{\partial \xi^{p_1} \partial \eta^{p_2} \partial \zeta^{p_3}}\right) \prod_{i=1}^{p_3} \frac{\partial^{s_i} F}{\partial \xi^{s_i'} \partial \eta^{s_i''}} \qquad \begin{pmatrix} 1 \leq p \leq l \\ p_2 + p_2 + \Sigma s_i \leq l \end{pmatrix}.$$

The lemma will be proved if we show that

$$\varphi(\xi,\eta) \in H(0, c\,A, \lambda).$$

The derivative $\dfrac{\partial^p f}{\partial \xi^{p_1} \partial \eta^{p_2} \partial \zeta^{p_3}}$ is H-continuous in (D_i); its limit is an H-continuous function on (S). Therefore,

$$\left(\frac{\partial^p f}{\partial \xi^{p_1} \partial \eta^{p_2} \partial \zeta^{p_3}}\right) \in H(0, c_1 A, \lambda) \quad \text{in} \quad (\Lambda_0).$$

Moreover,

$$\frac{\partial^{s_i} F}{\partial \xi^{s_i'} \partial \eta^{s_i''}} \in H(0, B, \lambda) \quad \text{in} \quad (\Lambda_0),$$

so that

$$\varphi(\xi,\eta) \in H(0, c_2 A, \lambda) \quad \text{in } (\Lambda_0);$$

here c_2 depends only on B and l. Since each derivative of (f) up to order l is the sum of a finite number of such functions $\varphi(\xi,\eta)$, it follows that the absolute values of all derivatives of (f) up to order l have an upper bound of the form $c_3 A$. The lemma is herewith established.

§19. The Potential of the Simple and Double Layer for a Surface L_k

We denote by $V[\mu]$ the potential of a simple layer with density μ and by $W[\mu]$ that of a double layer with density μ. Let $\mu \in H(0,A,\lambda)$, i.e., let μ be H-continuous on (S). Theorem 3 of §3 then assumes the following form:

Theorem 1. *If $\mu \in H(0,A,\lambda)$ on (S) and $(S) \in L_1(B,\lambda)$, then*

$$W[\mu] \in H(0, cA, \lambda') \quad \text{in } (D_i) \text{ and in } (D_e),$$

where λ' is an arbitrary positive number satisfying the inequality $\lambda' < \lambda$ and c depends only on B and on the choice of λ'.

We may similarly reformulate the theorem in §7:

Theorem 2. *If $\mu \in H(0,A,\lambda)$ on (S) and $(S) \in L_1(B,\lambda)$, then*

$$V[\mu] \in H(1, cA, \lambda') \quad \text{in } (D_i) \text{ and in } (D_e).$$

Let us also recall the results of §§8 and 10.

The result of §8 reads: If μ is differentiable on (S) and if (S) has continuous curvature, then

$$\frac{\partial V[\mu]}{\partial x} = V[D_x \mu - \mu K \cos(Nx)] + W[\mu \cos(Nx)]. \tag{69}$$

The result of §10 reads: If μ is differentiable on (S) and if $(S) \in L_1$, then

$$\begin{aligned} \frac{\partial W[\mu]}{\partial x} = {} & \frac{\partial}{\partial y} V[\cos(Ny) D_x \mu - \cos(Nx) D_y \mu] \\ & + \frac{\partial}{\partial z} V[\cos(Nz) D_x \mu - \cos(Nx) D_z \mu]. \end{aligned} \tag{70}$$

Making use of Theorems 1 and 2 and formulas (69) and (70), we shall now prove the following theorems.

Theorem 3. *If $(S) \in L_{k+1}(B,\lambda)$ $(k \geqq 0)$ and if $\mu \in H(l,A,\lambda)$ on (S) $(0 \leqq l \leqq k+1)$, then*

$$W[\mu] \in H(l, cA, \lambda') \quad \text{in } (D_i) \text{ and in } (D_e).$$

Theorem 4. *If $(S) \in L_{k+1}(B,\lambda)$ $(k \geqq 0)$ and if μ_1 $H(l_1,A,\lambda)$ on (S) $(0 \leqq l_1 \leqq k)$ then*

$$V[\mu_1] \in H(l_1 + 1, cA, \lambda') \quad \text{in } (D_i) \text{ and in } (D_e).$$

Proof. We prove both theorems together. We first note that Theorem 1 is a special case Theorem 3 ($l = 0$, $k = 0$), and Theorem 2 is a special case of Theorem 4 ($l_1 = 0$, $k = 0$). Since a surface of class L_{k+1} belongs to the class L_1, Theorems 3 and 4 have been proved for $l = 0$, $l_1 = 0$, $k \geqq 1$. It remains to consider the case $l \neq 0$, $l_1 \neq 0$, $k \geqq 1$.

The following two assertions are true:

1. If Theorem 4 is true for some l_1 with $0 \leqq l_1 \leqq k$, then Theorem 3 is true for $l = l_1 + 1$.
2. If Theorem 4 is true for some l_1' with $0 \leqq l_1' \leqq k - 1$ and Theorem 3 is true for $l = l_1' + 1$, then Theorem 4 holds for $l_1 = l_1' + 1 \leqq k$.

Making use of these assertions, the proofs of which we shall give directly, we now prove Theorems 3 and 4.

Indeed, since Theorem 4 is proved for $l_1 = 0$, the validity of Theorem 3 for $l = 1$ follows from the first assertion. From Theorem 2, Theorem 3 for $l = 1$, and the second assertion then follows the validity of Theorem 4 for $l_1 = 1$. Thecrem 3 then follows for $l = 2$ from the first assertion, and Theorem 4 for $l_1 = 2$ from the second assertion. One continues these considerations as long as $l_1 \leqq k$ and $l \leqq k + 1$. It now remains to prove assertions 1 and 2.

We first prove the first assertion. Let $\mu \in H(l_1 + 1, A, \lambda)$ $(0 \leqq l_1 \leqq k)$. From Lemmas 1 and 2 of 18 we conclude that

$$\cos(Nx),\ \cos(Ny),\ \cos(Nz) \in H(k, c_1, \lambda),$$

and hence, since $l_1 \leqq k$,

$$\cos(Nx),\ \cos(Ny),\ \cos(Nz) \in H(l_1, c_2, \lambda).$$

Since $l_1 + 1 \leqq k + 1$, we have moreover that

$$D_x\mu,\ D_y\mu,\ D_z\mu \in H(l_1, c_3 A, \lambda).$$

Hence,

$$\cos(Ny) D_x\mu,\ \cos(Nx) D_y\mu,\ \ldots \in H(l_1, c_4 A, \lambda);$$

since Theorem 4 is true for l_1 by hypothesis, we conclude:

$$V[\cos(Ny) D_x\mu - \cos(Nx) D_y\mu],$$
$$V[\cos(Nz) D_x\mu - \cos(Nx) D_z\mu] \in H(l_1 + 1, c_5 A, \lambda');$$

from formula (70) it now follows that

$$\frac{\partial W[\mu]}{\partial x} \in H(l_1, 2c_5 A, \lambda'). \tag{71}$$

It is easy to see that there exists a constant $c_6 > 0$ such that

$$|W[\mu]| < c_6 A. \tag{72}$$

Indeed, $W[\mu]$ is harmonic in the interior of (D_i) and in the interior of (D_e), such that the maximum of $|W[\mu]|$ is assumed on (S). However, the latter is no greater than

$$A\left[2\pi + \max_{M_0 \in (S)} \int_{(S)} \frac{|\cos(r_{10} N_1)|}{r_{10}^2}\, d\sigma_1\right],$$

Denoting the larger of the two numbers $2c_5$ and c_6 by c, from (71) and (72) we conclude that

$$W[\mu] \in H(l_1+1, cA, \lambda'); \tag{73}$$

the first assertion is herewith established.

We now prove the second assertion. Let $\mu_1 \in H(l_1' + 1, A, \lambda)$ $(0 \leqq l_1' \leqq k-1)$. Since $l_1' + 1 \leqq k$, it follows that $\cos(Nx) \in H(l_1' + 1, c, \lambda)$ and hence that $\mu_1 \cos(Nx) \in H(l_1' + 1, c_1 A, \lambda)$. Since Theorem 3 is valid for $l = l_1' + 1$ by hypothesis, we conclude:

$$W[\mu_1 \cos(Nx)] \in H(l_1'+1, c_2 A, \lambda') \quad \text{in } (D_i) \text{ and in } (D_e). \tag{74}$$

Further,

$$D_x \mu_1 \in H(l_1', c_3 A, \lambda).$$

We assume that $k \geqq 1$; then $k + 1 \geqq 2$, so that (S) has continuous curvature and formula (69) is applicable. Since

$$\cos(Nx),\ \cos(Ny),\ \cos(Nz) \in H(k, c, \lambda),$$

we then have for the mean curvature K of (S) (introduced in I, §3):

$$K = D_x \cos(Nx) + D_y \cos(Ny) + D_z \cos(Nz) \in H(k-1, c_4, \lambda).$$

Since $l_1' \leqq k - 1$ therefore

$$\cos(Nx),\ \cos(Ny),\ \cos(Nz),\ K \in H(l_1', c_5, \lambda).$$

Hence,

$$\mu_1 K \cos(Nx) \in H(l_1', c_6, \lambda).$$

From this it follows that

$$D_x \mu_1 - \mu_1 K \cos(Nx) \in H(l_1', c_7, \lambda).$$

From the assumed validity of Theorem 4 for l_1' we conclude:

$$V[D_x \mu_1 - \mu_1 K \cos(Nx)] \in H(l_1'+1, c_8 A, \lambda') \quad \text{in } (D_i) \text{ and in } (D_e).$$

From this using (69) and (74) we find that

$$\frac{\partial V[\mu_1]}{\partial x} \in H(l_1, c_9 A, \lambda') \text{ in } (D_i) \text{ and in } (D_e),\ l_1 = (l_1' + 1). \tag{75}$$

Since $V[\mu_1]$ is harmonic in the interior of (D_i) and in the interior of (D_e), $|V[\mu_1]|$ assumes its maximum on (S). We have therefore:

$$|V[\mu_1]| \leq A \operatorname*{Max}_{M_0 \in (S)} \int_{(S)} \frac{d\sigma_1}{r_{10}} = c_{10} A. \tag{76}$$

Denoting the larger of the two numbers c_9 and c_{10} by c, we find from (75) and (76) that

$$V[\mu_1] \in H(l_1 + 1, cA, \lambda'), \tag{77}$$

wherewith assertion 2 is established. Theorems 3 and 4 have now been completely proved.

§20. The Newtonian Potential in a Region Bounded by a Surface L_k

We denote the Newtonian potential with density μ by $P[\mu]$. The theorem of 16 may then be formulated in the following manner:

Theorem 1. *If $\mu \in H(0,A,\lambda)$ in (D_i) and (D_i) is bounded by a surface $(S) \in L_1(B,\lambda)$, then*

$$P[\mu] \in H(2, cA, \lambda').$$

Formula (61) of §17 for the first derivative of the Newtonian potential with differentiable density may be written in the form

$$\frac{\partial P[\mu]}{\partial x} = P\left[\frac{\partial \mu}{\partial x}\right] - V[\mu \cos(Nx)] \tag{79}$$

We prove the following theorem:

Theorem 2. *If $(S) \in L_{k+1}(B,\lambda)$ and $\mu \in H(l,A,\lambda)$ $(0 \leqq l \leqq k)$, then*

$$P[\mu] \in H(l+2, cA, \lambda') \quad \text{in } (D_i) \text{ and in } (D_e),$$

where λ' is an arbitrary positive number in the interval $0 < \lambda' < \lambda$ and c depends only on B and on the choice of λ'.

Proof. Theorem 2 includes Theorem 1 as a special case ($l = k = 0$), so that Theorem 2 is proved for the case $l = 0$, $k = 0$.

Suppose now that $k \geqq 1$. We wish to show that the theorem holds for $\mu \in H(l,A,\lambda)$ $(1 \leqq l \leqq k)$ under the hypothesis that it is true for $\mu \in H(l-1,A,\lambda)$ in (D_i). Now if in (D_i) $\mu \in H(l,A,\lambda)$, then

$$\frac{\partial \mu}{\partial x} \in H(l-1, A, \lambda) \text{ in } (D_i).$$

Since the theorem is true for $\mu \in H(l-1,A,\lambda)$ by hypothesis, we conclude that

$$P\left[\frac{\partial \mu}{\partial x}\right] \in H(l+1, c_1 A, \lambda') \text{ in } (D_i) \text{ and in } (D_e). \tag{80}$$

Since $\mu \in H(l,A,\lambda)$ in (D_i), from Lemma 3 of §18 $(\mu) \in H(l,c_2A,\lambda)$ on (S). Further, from Lemma 1 of §18

$$\cos(Nx) \in H(k, c_3, \lambda),$$

and hence since $l \leqq k$ $\quad \cos(Nx) \in H(l, c_4, \lambda)$.

It then follows that

$$\mu \cos(Nx) \in H(l, c_5 A, \lambda) \quad \text{on} \quad (S).$$

Using Theorem 4 of §19, we now obtain:

$$V[\mu \cos(Nx)] \in H(l+1, c_6 A, \lambda') \quad \text{in } (D_i) \text{ and in } (D_e). \tag{81}$$

From formulas (79), (80), and (81) we conclude:

$$\frac{\partial P[\mu]}{\partial x} \in H(l+1, c_7 A, \lambda'). \tag{82}$$

It is easily seen that there exists a constant $c_8 > 0$ such that

$$\text{in } (D_i) \text{ and in } (D_e) \quad |P[\mu]| < c_8 A \tag{83}$$

Indeed, $P[\mu]$ is a harmonic function in the interior of (D_e), so that $|P[\mu]|$ assumes its maximum either on (S) or in the interior of (D_i). The latter is however no greater than

$$A \cdot \operatorname{Max} \int_{(D_i)} \frac{d\tau}{r_{10}} = c_8 A \quad (M_0 \text{ on } (S) \quad \text{or in } (D_i)).$$

Denoting the larger of the constants c_7 and c_8 by c, we conclude from (82) and (83) that

$$P[\mu] \in H(l+2, cA, \lambda')$$

in the case that $\mu \in H(l-1, A, \lambda)$ in (D_i).

Since Theorem 2 has been proved for $\mu \in H(0,A,\lambda)$, it holds for $\mu \in H(l,A,\lambda)$ with $l = 1, 2, \ldots, k$.

§21. The Values on a Surface L_k of the Potential of a Double Layer and of the Normal Derivative of the Potential of a Simple Layer

Under the hypothesis of the boundedness of the density μ it was shown in §§3 and 5 that the potential of the double layer and the normal derivative of the potential of the simple layer are H-continuous when considered as functions of the points of (S). The following two theorems were proved there.

Theorem 1. *If* $(S) \in L_1$ *and if* $|\mu| < A$ *on* (S), *then* $\overline{W}[\mu] \in H(0,cA,\lambda')$ *on* (S).

Theorem 2. *If* $(S) \in L_1$ *and if* $|\mu| < A$ *on* (S), *then* $\dfrac{dV[\mu]}{dn} \in H(0,cA,\lambda')$ *on* (S).

Moreover, the following theorems are true:

Theorem 3. *If* $(S) \in L_{l+2}$ *and if* $\mu \in H(l,A,\lambda)$ $(l \geqq 0)$ *on* (S), *then* $\overline{W}[\mu] \in H(l+1,cA,\lambda')$ *on* (S).

Theorem 4. *If* $(S) \in L_{l+2}(B,\lambda)$ *and if* $\mu \in H(l,A,\lambda)$ $(l \geqq 0)$ *on* (S), *then* $\dfrac{dV[\mu]}{dn} \in H(l+1,cA,\lambda')$ on (S).

The proof of Theorems 3 and 4 is rather involved; it may be found in Appendix IV.

§22. Remarks on the Potentials of Class $C^{(k)}$

We shall say that *a function* $f(x,y,z) = f(M)$ *defined in the region* (D) *belongs to the class* $C^{(k)}(A)$ *if* f *has continuous and bounded derivatives of order* $k (k \geqq 1)$ *in* (D) *and if these derivatives and the function itself are less than or equal to* A *in absolute value.*

If the region (D) is bounded by a LYAPUNOV surface, then the derivatives of f of order $k-1$ are H-continuous in (D); hence,

$$f \in H(k-1, cA, 1),$$

where c is some constant which depends only on the region. If, on the other hand, $f \in H(k,A,\lambda)$, then $f \in C^{(k)}(A)$ independent of the character of the region (D). If f is continuous and bounded in (D), then we shall write $f \in C^{(0)}(A)$.

We give an analogous definition for a function of two variables. We shall say that *a function* μ *on* (S) *belongs to the class* $C^{(k)}(A)$ *if the function* $\mu(\xi,\eta)$ *belongs to the class* $C^{(k)}(A)$ *in* (Λ_0) *and* A *is independent of the choice of the point* M_0 *on* (S).

From the theorems of §§19, 20, and 21 we obtain the following theorems for functions and surfaces of class $C^{(k)}$.[2]

Theorem 1. *If* $(S) \in C^{(k+1)} (k \geqq 1)$ *and if* $\mu \in C^{(l)}(A)$ $(1 \leqq l \leqq k+1)$ *on* (S), *then*

$$W[\mu] \in C^{(l-1)}(cA) \quad in\ (D_i) \text{ and } in\ (D_e).$$

Moreover, if $\mu \in C^{(0)}(A)$, then

$$W[\mu] \in C^{(0)}(cA) \quad in\ (D_i) \text{ and } in\ (D_e).$$

The assertion for $l \geqq 1$ follows from Theorem 3 of §19, since from $\mu \in C^{(l)}$

[2] Theorems 1, 2, and 3 of this section were proved by E. SCHMIDT in the paper *Remarks on Potential Theory*, Mathematische Abhandlungen, HERMANN AMANDUS SCHWARZ zu seinem fünfzigjährigen Doktorjubiläum, Berlin, 1914. In this paper use was made of the method of complete induction which we applied in §§19 and 20.

one concludes that $\mu \in H(l-1, cA, 1)$; for $l = 0$ the theorem follows from the fact that the potential of a double layer with continuous density is continuous in (D_i) and in (D_e), and its absolute value is no greater than a number of the form cA.

Theorem 2. *If* $(S) \in C^{(k+1)} (k \geqq 1)$ *and if* $\mu \in C^{(l)} (0 \leqq l \leqq k)$, *then*

$$V[\mu] \in C^{(l)}(cA) \quad \textit{in } (D_i) \text{ and } \textit{in } (D_e).$$

For $l \geqq 1$ the theorem follows from Theorem 4 of §19; for $l = 0$ it follows from the fact that the potential of a simple layer with bounded density is continuous and is not greater than a number cA in absolute value.

One can easily find examples of functions $\mu \in C^{(l)}$ for which the derivatives of order l of $W[\mu]$ or of order $l+1$ of $V[\mu]$ are unbounded in (D_i) or in (D_e), so that $W[\mu]$ does not belong to the class $C^{(l)}$ or $V[\mu]$ does not belong to the class $C^{(l+1)}$.

Theorem 3. *If* $(S) \in C^{(k+1)} (k \geqq 1)$ *and if* $\mu \in C^{(l)}(A)$ $(0 \leqq l \leqq k)$ in (D_i), *then*

$$P[\mu] \in C^{(l+1)}(cA) \quad \textit{in } (D_i) \text{ and } \textit{in } (D_e).$$

For $l \geqq 1$ the theorem is a consequence of Theorem 2 of §20; for $l = 0$ it follows from the theorem which says that a Newtonian potential with bounded density is H-continuous everywhere.

Finally, we obtain from the theorems in §21:

Theorem 4. *If* $(S) \in C^{(l+2)}(B)$ *and if* $\mu \in C^{(l)}(A)$ $(l \geqq 0)$ *on* (S), *then on* (S)

$$\overline{W}[\mu] \quad \text{and} \quad \frac{d\overline{V}[\mu]}{dn} \in C^{(l)}(cA).$$

§23. The Potential of the Simple and Double Layer with Summable Density

Before we begin the study of the properties of the potential of a simple layer with summable density we shall assemble without proof those properties of summable functions and of the LEBESGUE integral which will subsequently be used. We shall here not define the LEBESGUE integral and summable functions, but rather refer the reader to the texts by V. I. SMIRNOV or I. P. NATANSON.[3]

[3] SMIRNOV, V. I., *A Course of Higher Mathematics* (English translation). Vol. V, Pergamon Press, 1964.
NATANSON, I. P., *Theory of Functions of a Real Variable* (English translation). Vol. I, Frederick Ungar, New York, 1961.

In order to have a specific case in mind, we shall formulate those properties of summable functions necessary for our considerations for functions defined and summable on a surface (S) in the form in which they will subsequently be used.

1. *The inequality for the absolute value of an integral.* Let μ be summable on (S), and let f be continuous and less than or equal to A in absolute value, $|f| \leqq A$. Then μf is summable, and we have:

$$\left|\int_{(S)} \mu f \, d\sigma\right| < A \int_{(S)} |\mu| \, d\sigma .$$

From this it follows immediately that if a function $f_h(M)$ defined on (S) is continuous and depends on the parameter h and if as $h \to 0$ it converges uniformly to the continuous function $f(M)$, then

$$\lim_{h \to 0} \int_{(S)} \mu f_h \, d\sigma = \int_{(S)} \mu f \, d\sigma .$$

2. *The property of absolute continuity of the* LEBESGUE *integral.* If μ is summable on (S), then for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int_{(\delta)} |\mu| \, d\sigma < \varepsilon ,$$

where (δ) is the subregion of the surface (S) contained in the interior of a sphere of radius δ about an arbitrary point of (S).

3. *The* LEBESGUE *points of a summable function.* Let M_0 be a certain point of (S) and $0 < \delta \leqq \frac{d}{2}$, where d is the radius of the LYAPUNOV sphere. We denote by (δ, M_0) the subregion of (S) contained in the LYAPUNOV sphere about M_0 which projects into the circle of radius δ about M_0 in the tangent plane at M_0. If now μ is a summable function on (S), then M_0 is called a LEBESGUE point of μ if the relation

$$\lim_{\delta \to 0} \frac{1}{\pi \delta^2} \int_{(\delta, M_0)} |\mu - \mu_0| \, d\sigma = 0 ,$$

holds where μ_0 is the value of μ at the point M_0.

We denote the set of LEBESGUE points of the function μ by Q_μ. It is well-known that the set of points which are not LEBESGUE points has measure zero, i.e., Measure $(S - Q_\mu) = 0$.

4. *The* FUBINI *theorem.* Let $F(1,2)$ be a function of the two points M_1 and M_2 of the surface (S). As a function of M_2 let $F(1,2)$ be summable for all M_1 except possibly for a set of measure zero, and likewise as a function of M_1 let it be summable for all M_2 with the possible exception of a set of measure zero. We consider the integral

$$\varphi(1) = \int\limits_{(S_2)} |F(1, 2)|\, d\sigma_2,$$

which is a function of the point M_1; this integral can diverge for points M_1 forming a set of measure zero.

From the FUBINI theorem on multiple LEBESGUE integrals we can draw the following conclusion:

If the function $\varphi(1)$ is summable on (S), then the integral

$$\psi(2) = \int\limits_{(S_1)} F(1, 2)\, d\sigma_1$$

exists for all points M_2 excepting those of a set of measure zero and is a summable function on (S). Moreover, we have:

$$\int\limits_{(S_1)} \left[\int\limits_{(S_2)} F(1, 2)\, d\sigma_2\right] d\sigma_1 = \int\limits_{(S_2)} \left[\int\limits_{(S_1)} F(1, 2)\, d\sigma_1\right] d\sigma_2.$$

Functions summable in (D) possess analogous properties.

We now turn to the investigation of a simple-layer potential the density μ of which is summable on (S); the surface (S) is hereby assumed to be a LYAPUNOV surface. We have:

$$V(0) = \int\limits_{(S_1)} \mu(1) \frac{d\sigma_1}{r_{10}}. \tag{84}$$

If the point M_0 does not lie on (S), then $\frac{1}{r_{10}}$ is a continuous function of the point M_1 of (S), and the integral (84) is convergent. Moreover, the above inequality for the absolute value of the integral implies that $V(0)$ is a continuous function which has continuous derivatives of arbitrary order at any point not on (S). It moreover satisfies the LAPLACE equation, and the function $V(0)$ and its derivatives tend to zero as the point M_0 goes to infinity. It thus follows that $V(0)$ is harmonic in any region which together with its boundary is contained in (D_i) or (D_e). We shall now investigate the behavior of $V(0)$ as the point M_0 moves toward the surface (S).

Theorem 1. *Let μ be summable on (S). Then the integral (84) is convergent if the point M_0 on (S) does not belong to a certain set of measure zero. Moreover, $V(0)$ is a summable function on (S).*

Proof. We put

$$F(0, 1) = \frac{\mu(1)}{r_{10}}$$

and consider the integral

$$\varphi(1) = \int\limits_{(S_0)} |F(0, 1)|\, d\sigma_0 = \int\limits_{(S_0)} \frac{|\mu(1)|}{r_{10}}\, d\sigma_0 = |\mu(1)| \int\limits_{(S_0)} \frac{d\sigma_0}{r_{10}}.$$

Since the integral

$$\int\limits_{(S_0)} \frac{d\sigma_0}{r_{10}}$$

is a continuous and bounded function of the point M_1 on (S), the function $\varphi(1)$, which is the product of this function and the summable function $|\mu(1)|$, is a summable function of M_1 on (S). From the FUBINI theorem it now follows that the integral (84) converges if M_0 does not belong to a certain set of measure zero which depends on the function μ. We conclude moreover that $V(0)$ is summable on (S). The set of points for which the integral (84) converges we denote by Q'_μ.

Theorem 2. *Let the point M_0 belong to the set Q'_μ, and let M_2 be a point lying outside (S) on the normal to (S) at the point M_0. Then*

$$\lim_{M_2 \to M_0} V(2) = V(0).$$

Proof. Since M_0 belongs to the set Q'_μ, $\dfrac{\mu(1)}{r_{10}}$ is summable on (S). From the absolute continuity of the LEBESGUE integral of a summable function it follows that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\int_{(\delta)} \left| \frac{\mu(1)}{r_{10}} \right| d\sigma_1 < \varepsilon. \tag{85}$$

We denote by ϱ the projection of r_{10} onto the tangent plane at the point M_0. It is clear that for all points M_1 of (δ)

$$\frac{r_{10}}{2} < \varrho < r_{12}$$

independent of the location of the point M_2 on the normal N_0 at the point M_0. From this it follows that

$$\frac{1}{r_{12}} < \frac{2}{r_{10}}. \tag{86}$$

Whence

$$\int_{(\delta)} \left| \frac{\mu(1)}{r_{12}} \right| d\sigma_1 < 2\varepsilon \tag{87}$$

independent of the location of the point M_2 on N_0. We have:

$$\begin{aligned} V(2) - V(0) = &\int_{(\delta)} \mu(1) \frac{d\sigma_1}{r_{12}} - \int_{(\delta)} \frac{\mu(1)}{r_{10}} d\sigma_1 \\ &+ \int_{(S-\delta)} \mu(1) \left[\frac{1}{r_{12}} - \frac{1}{r_{10}} \right] d\sigma_1 . \end{aligned} \tag{88}$$

The absolute value of the difference of the first two integrals on the right-hand side of (88) is no greater than 3ε. In a certain neighborhood of M_0 the last integral in (88) is a continuous function of M_2 and tends to zero as $M_2 \to M_0$. It is therefore possible to find an $h > 0$ such that this integral is less than ε whenever $r_{20} < h$. Hence, from (88) it follows that

$$|V(2) - V(0)| < 4\varepsilon,$$

for $r_{20} < h$. Theorem 2 is herewith proved.

As we have seen, the integrals

$$\int\limits_{(S_0)} \frac{|\cos(r_{10}N_0)|}{r_{10}^2} d\sigma_0 \qquad \text{and} \qquad \int\limits_{(S_0)} \frac{|\cos(r_{10}N_1)|}{r_{10}^2} d\sigma_0$$

are convergent and represent continuous and bounded functions of M_0 on (S). Just as in Theorem 1 of this section, it now follows that if μ is summable on (S), then the integrals

$$\left.\begin{aligned} \frac{dV}{dn} &= \int\limits_{(S_1)} \mu(1) \frac{\cos(r_{10}N_0)}{r_{10}^2} d\sigma_1 \\ \text{and} \qquad \overline{W}(0) &= \int\limits_{(S_1)} \mu(1) \frac{\cos(r_{10}N_1)}{r_{10}^2} d\sigma_1 \end{aligned}\right\} \tag{89}$$

are convergent whenever the point M_0 on (S) does not belong to a certain set of measure zero; these integrals are summable functions of M_0 on (S). We denote the set of points for which the integrals (89) converge by Q_μ''.

Lemma. *If M_0 is a* Lebesgue *point of the summable function μ on (S), then the integral*

$$\int\limits_{(\delta, M_0)} \frac{|\mu - \mu_0|}{\varrho^{2-\lambda}} d\sigma \qquad (0 < \lambda \le 1) \tag{90}$$

is convergent and has limit zero as $\delta \to 0$. Moreover, the integral

$$|z| \cdot \int\limits_{(\delta, M_0)} \frac{|\mu - \mu_0|}{(\sqrt{\varrho^2 + z^2})^3} d\sigma \tag{91}$$

tends uniformly to zero as $\delta \to 0$ for all nonzero values of z.

Proof. We introduce the function

$$\psi(\varrho) = \int\limits_{(\varrho, M_0)} |\mu - \mu_0| \, d\sigma = \int\limits_0^\varrho \left[\varrho_1 \int\limits_0^{2\pi} |\mu - \mu_0| \frac{d\varphi}{|\cos(NN_0)|} \right] d\varrho_1 .$$

Since M_0 is a Lebesgue point, $\psi(\varrho) = \varrho^2 \varepsilon(\varrho)$ with $\lim\limits_{\varrho \to 0} \varepsilon(\varrho) \to 0$. Moreover, $\psi(\varrho)$ is an absolutely continuous function of ϱ and

$$\psi'(\varrho) = \varrho \int\limits_0^{2\pi} |\mu - \mu_0| \frac{d\varphi}{|\cos(NN_0)|} \cdot$$

In the following $f(\varrho)$ shall denote either the function $\varrho^{\lambda-2}$ or the function $|z|(\varrho^2 + z^2)^{-3/2}$. Since $\lim\limits_{\varrho \to 0} \psi(\varrho) f(\varrho) = 0$, the two integrals to be investigated assume the following form:

$$\int\limits_{(\delta, M_0)} |\mu - \mu_0|\, f(\varrho)\, d\sigma = \int\limits_0^{\delta} f(\varrho) \left[\varrho \int\limits_0^{2\pi} |\mu - \mu_0| \frac{d\varphi}{|\cos(N N_0)|}\right] d\varrho$$

$$= \int\limits_0^{\delta} f(\varrho)\, \psi'(\varrho)\, d\varrho = \psi(\varrho) f(\varrho)\Big|_0^{\delta} - \int\limits_0^{\delta} \psi(\varrho) f'(\varrho)\, d\varrho$$

$$= \varepsilon(\delta)\, \delta^2 f(\delta) - \int\limits_0^{\delta} \varepsilon(\varrho)\, \varrho^2 f'(\varrho)\, d\varrho .$$

In the case $f(\varrho) = \varrho^{\lambda-2}$

$$0 < -\int\limits_0^{\delta} \varepsilon(\varrho)\, \varrho^2 f'(\varrho)\, d\varrho = (2-\lambda) \int\limits_0^{\delta} \varepsilon(\varrho)\, \varrho^{\lambda-1} d\varrho$$

$$= (2-\lambda)\, \varepsilon(\varrho_1) \int\limits_0^{\delta} \varrho^{\lambda-1} d\varrho = \frac{(2-\lambda)\, \varepsilon(\varrho_1)}{\lambda}\, \delta^{\lambda} \qquad (0 < \varrho_1 < \delta) .$$

Hence,

$$\int\limits_{(\delta, M_0)} \frac{|\mu - \mu_0|}{\varrho^{2-\lambda}}\, d\sigma = \delta^{\lambda} \left[\varepsilon(\delta) + \frac{2-\lambda}{\lambda}\, \varepsilon(\varrho_1)\right] \qquad (0 < \varrho_1 < \delta) ,$$

so that the lemma is proved for the integral (90).

In the case $f(\varrho) = |z| \cdot (\varrho^2 + z^2)^{-3/2}$

$$f'(\varrho) = -3\, |z| \cdot \varrho (\varrho^2 + z^2)^{-5/2}$$

and hence

$$0 < -\int\limits_0^{\delta} \varepsilon(\varrho)\, \varrho^2 f'(\varrho)\, d\varrho = 3\, |z| \int\limits_0^{\delta} \frac{\varepsilon(\varrho)\, \varrho^3}{(\varrho^2 + z^2)^{5/2}}\, d\varrho < 3\, |z| \int\limits_0^{\delta} \frac{\varepsilon(\varrho)}{\varrho^2 + z^2}\, d\varrho$$

$$= 3\, \varepsilon(\varrho_1) \int\limits_0^{\delta} \frac{|z|}{\varrho^2 + z^2}\, d\varrho = 3\, \varepsilon(\varrho_1) \operatorname{arctg} \frac{\varrho}{|z|}\Big|_{\varrho=0}^{\delta} < \frac{3\pi}{2}\, \varepsilon(\varrho_1) \qquad (0 < \varrho_1 < \delta) .$$

It follows further that

$$|z| \int\limits_{(\delta, M_0)} \frac{|\mu - \mu_0|}{(\sqrt{\varrho^2 + z^2})^3}\, d\sigma < \varepsilon(\delta)\, \frac{\delta^2 |z|}{(\sqrt{\delta^2 + z^2})^3} + \frac{3\pi}{2}\, \varepsilon(\varrho_1) < \varepsilon(\delta) + \frac{3\pi}{2}\, \varepsilon(\varrho_1) .$$

From this now follows the assertion of the lemma for the integral (91).

If M_0 is a Lebesgue point of the function μ from the convergence of integral (90) one can conclude the convergence of integrals (84) and (89):

$$\int\limits_{(S_1)} \mu(1) \frac{\cos(r_{10} N_1)}{r_{10}^2}\, d\sigma_1 = \int\limits_{(S_1 - (\delta, M_0))} \mu(1) \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1$$

$$+ \mu_0 \int\limits_{(\delta, M_0)} \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1 + \int\limits_{(\delta, M_0)} (\mu - \mu_0) \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1 .$$

The convergence of the first two integrals on the right-hand side is clear, while the convergence of the third integral for a LEBESGUE point of μ follows from the fact that in (δ, M_0) the estimate

$$\left|\frac{\cos(r_{10} N_0)}{r_{10}^2}\right| < \frac{b}{\varrho^{2-\lambda}} \tag{92}$$

holds. The proof for the integral (84) and for the second integral in (89) is analogous.

It follows from what has been said that the set Q_μ of the LEBESGUE points of the function μ is contained in the sets Q'_μ and Q''_μ for which the integrals (84) and (89) converge.

Let the point M_2 lie on the normal N_0 at the point M_0 of (S) in either (D_i) or (D_e).

Theorem 3. *If M_0 belongs to the set Q_μ, then*

$$\frac{dV_i}{dn} = \lim_{M_2 \to M_0} \frac{dV(2)}{dn} = 2\pi\mu_0 + \frac{dV}{dn}, \qquad \text{for } M_2 \textit{ in } (D_i) \text{ and}$$

$$\frac{dV_e}{dn} = \lim_{M_2 \to M_0} \frac{dV(2)}{dn} = -2\pi\mu_0 + \frac{dV}{dn}, \qquad \text{for } M_2 \textit{ in } (D_e);$$

$\frac{dV}{dn}$ *is here defined by the first formula in* (89). *The limits are summable functions on* (S).

Proof. We have:

$$\begin{aligned}\frac{dV(2)}{dn} &= \int_{(S_1)} \mu(1) \frac{\cos(r_{12} N_0)}{r_{12}^2}\, d\sigma_1 = \int_{(S_1)} \mu(1) \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1 \\ &+ \mu_0 \left[\int_{(S_1)} \frac{\cos(r_{12} N_0)}{r_{12}^2}\, d\sigma_1 - \int_{(S_1)} \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1\right] \\ &+ \int_{(S_1)} (\mu(1) - \mu_0) \frac{\cos(r_{12} N_0)}{r_{12}^2}\, d\sigma_1 - \int_{(S_1)} (\mu(1) - \mu_0) \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1 .\end{aligned} \tag{93}$$

The difference in the square brackets on the right-hand side of (93) is the difference of the values of the derivatives of the potential of a simple layer with density $\mu = 1$ at the point M_2 in the direction of the normal N_0 and the direct value.[4] It follows that this difference has limit 2π for M_2 in (D_i) and -2π for M_2 in (D_e). The first summand on the right-hand side of (93) is equal to $\frac{dV}{dn}$.

[4] The term *direct value* refers to the value of a function considered as a function defined on a surface, e.g., $\overline{W}$ and $\frac{dV}{dn}$ as opposed to W_i, W_e, $\frac{dV_i}{dn}$, and $\frac{dV_e}{dn}$. (Trans.)

To prove the theorem it remains only to show that the difference of the two last integrals in (93) tends to zero. We write this difference in the form

$$\int\limits_{(\delta, M_0)} (\mu - \mu_0) \frac{\cos (r_{12} N_0)}{r_{12}^2} \, d\sigma_1 - \int\limits_{(\delta, M_0)} (\mu - \mu_0) \frac{\cos (r_{10} N_0)}{r_{10}^2} \, d\sigma_1 + \int\limits_{(S_1 - (\delta, M_0))} (\mu - \mu_0) \left[\frac{\cos (r_{12} N_0)}{r_{12}^2} - \frac{\cos (r_{10} N_0)}{r_{10}^2}\right] d\sigma_1 . \tag{94}$$

Let (ξ,η,ζ) be a local coordinate system with origin at M_0. The coordinates of the point M_1 we denote by (ξ,η,ζ); the point M_2 on the ζ axis shall have coordinates $(0,0,z)$.

Since the point $M_3(\xi,\eta,0)$ has distance $|\zeta|$ from M_1, it follows from considering the triangle $M_1M_2M_3$ that

$$\left| r_{21} - \sqrt{\varrho^2 + z^2} \right| \leq |\zeta| \leq b \varrho^{1+\lambda}.$$

Since $\varrho \leqq r_{21}$ we conclude that

$$\left| 1 - \frac{\sqrt{\varrho^2 + z^2}}{r_{21}} \right| \leq \frac{b \varrho^{1+\lambda}}{r_{21}} \leq b \varrho^{\lambda} ;$$

hence,

$$\frac{1}{r_{21}} \leq \frac{1 + b\varrho^{\lambda}}{\sqrt{\varrho^2 + z^2}} < \frac{1 + b\delta^{\lambda}}{\sqrt{\varrho^2 + z^2}}.$$

Further

$$|\cos (r_{12} N_0)| = \frac{|z - \zeta|}{r_{12}} < \frac{|z|(1 + b\,\delta^{\lambda})}{\sqrt{\varrho^2 + z^2}} + \frac{b\,\varrho^{1+\lambda}}{\varrho}.$$

From the last two inequalities it follows that

$$\frac{|\cos (r_{12} N_0)|}{r_{12}^2} < \frac{|z|}{(\sqrt{\varrho^2 + z^2})^3} (1 + b\,\delta^{\lambda})^3 + \frac{b}{\varrho^{2-\lambda}} (1 + b\delta^{\lambda})^2.$$

From the lemma we may now conclude that the first integral in (94) has limit zero as $\delta \to 0$; similarly, from inequality (92) the second integral also has limit zero as $\delta \to 0$. From this it follows that given $\varepsilon > 0$ one can find a $\delta > 0$ such that the absolute value of the difference of the first two integrals in (94) is no greater than ε. For fixed δ the third integral in (94) tends to zero as $M_2 \to M_0$, so that one can find an $h > 0$ such that the absolute value of the third integral is less than ε for $r_{02} < h$. Thus, the absolute value of (94) is less than 2ε for $r_{02} < h$, whence it follows that the difference of the last two integrals in (93) tends to zero as $M_2 \to M_0$. Since μ and $\frac{dV}{dn}$ are summable functions on (S), $\frac{dV_i}{dn}$ and $\frac{dV_e}{dn}$ also possess these properties. This completes the proof of the theorem.

We now consider the potential of the double layer with the summable density μ on (S):

$$W(0) = \int_{(S_1)} \mu(1) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 .$$

If the point M_0 lies off (S), then $W(0)$ is continuous, has continuous derivatives of arbitrary order, and satisfies the LAPLACE equation. We have moreover seen that the integral (95) converges if M_0 belongs to the set Q_μ'' on (S). Let the point M_2 lie in (D_i) or in (D_e) on the normal N_0.

Theorem 4. *If M_0 belongs to the set Q_μ, then*

$$\begin{aligned} W_i &= \lim_{M_2 \to M_0} W(2) = 2\pi\mu_0 + \overline{W}(0), && \text{for } M_2 \text{ in } (D_i) \text{ and} \\ W_e &= \lim_{M_2 \to M_0} W(2) = -2\pi\mu_0 + \overline{W}(0), && \text{for } M_2 \text{ in } (D_e)\,; \end{aligned} \tag{95}$$

$\overline{W}(0)$ is here defined by the second formula in (89). The functions $W_i(0)$ and $W_e(0)$ are summable on (S).

Proof. In place of equation (93) we now have:

$$W(2) = \overline{W}(0) \pm 2\pi\mu_0 + \int_{(S_1)} (\mu - \mu_0) \frac{\cos(r_{12} N_1)}{r_{12}^2} d\sigma_1 - \int_{(S_1)} (\mu - \mu_0) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 , \tag{96}$$

and to prove the theorem it suffices to show that the difference of the two last integrals tends to zero as $M_2 \to M_0$. We note that in (δ, M_0) the inequalities

$$\begin{aligned} |\cos(r_{12} N_1) - \cos(r_{12} N_0)| &\leq (N_1 N_0) < E r_{10}^\lambda , \\ |\cos(r_{10} N_1) - \cos(r_{10} N_0)| &\leq (N_1 N_0) < E r_{10}^\lambda \end{aligned}$$

hold; hence,

$$\left| \int_{(\delta, M_0)} (\mu - \mu_0) \frac{\cos(r_{1k} N_1)}{r_{1k}^2} d\sigma_1 \right| < \left| \int_{(\delta, M_0)} (\mu - \mu_0) \frac{\cos(r_{1k} N_0)}{r_{1k}^2} d\sigma_1 \right| + 2^\lambda E \int_{(\delta, M_0)} \frac{|\mu - \mu_0|}{\varrho^{2-\lambda}} d\sigma_1 \quad (k = 0, 2)$$

As we have seen, the right-hand side of the last inequality has limit zero as $\delta \to 0$; hence, the left-hand side also tends to zero. Just as in Theorem 3 it now follows that the difference of the two last integrals in (96) goes to zero as

$M_2 \to M_0$. Since μ and $\overline{W}$ are summable functions on (S), W_i and W_e also have this property. The theorem is herewith proved.[5]

§24. The Newtonian Potential with Summable Density

Let $\mu(x,y,z) = \mu(M)$ be a summable function in the finite region (D). We consider the Newtonian potential with density μ:

$$P(0) = \int_{(D)} \frac{\mu(1)}{r_{10}} d\tau_1 . \tag{97}$$

If M_0 is an interior point of the region complementary to (D), then the integral (97) is convergent and represents a harmonic function in every region which together with its boundary is contained in the region complementary to (D).

If M_0 belongs to the region (D) or to its boundary, then the integral (97) may diverge. We shall show that it converges except for M_0 in a certain set of measure zero in (D). For this we put $F(0,1) = \dfrac{\mu(1)}{r_{10}}$ and consider the integral

$$\varphi(1) = \int_{(D)} |F(0,1)|\, d\tau_0 = \int_{(D)} \frac{|\mu(1)|}{r_{10}} d\tau_0 = |\mu(1)| \int_{(D)} \frac{d\tau_0}{r_{10}} .$$

Since the integral

$$\int_{(D)} \frac{d\tau_0}{r_{10}}$$

is a continuous and bounded function of M_1 in (D), the function $\varphi(1)$, which is the product of this function and the summable function $|\mu(1)|$, is summable in (D). From the Fubini theorem we conclude that the integral (97) converges except for M_0 in a certain set of measure zero. $P(0)$ is moreover summable in (D).

We shall now assume that μ and μ^2 are summable, i.e., μ belongs to L_2, and we shall denote by $\|\mu\|$ the *norm* of μ in L_2:

$$\|\mu\| = \left\{ \int_{(D)} |\mu|^2 d\tau \right\}^{1/2} .$$

Theorem. *If μ belongs to L_2, then $P[\mu]$ is H-continuous everywhere with*

$$|P(1) - P(2)| \leq C \|\mu\| \delta^{1/2} \tag{98}$$

where $\delta = r_{12}$ and $C = 4\sqrt{2\pi} + 2\sqrt{3\pi}$.

[5] The theorems of this section are contained in the paper by G. Fichera, *Teoremi di completezza sulla frontiera di un dominio per taluni sistemi di funzioni* (Annali di matematica pura ed applicata, t.XXVII, serie 4, 1948).

Proof. Putting μ equal to zero outside the region (D), we may assume that the integral (97) is extended over all of space. We then have:

$$\begin{aligned} P(1)-P(2) &= \int \frac{\mu(0)}{r_{10}}\,d\tau_0 - \int \frac{\mu(0)}{r_{20}}\,d\tau_0 \\ &= \int\limits_{r_{10}<2\delta} \frac{\mu(0)}{r_{10}}\,d\tau_0 - \int\limits_{r_{10}<2\delta} \frac{\mu(0)}{r_{20}}\,d\tau_0 + \int\limits_{r_{10}\geqq 2\delta} \mu(0)\left[\frac{1}{r_{10}}-\frac{1}{r_{20}}\right]d\tau_0 . \end{aligned} \tag{99}$$

From the BUNYAKOVSKII-SCHWARZ inequality we obtain:

$$\left|\int\limits_{r_{10}<2\delta} \frac{\mu(0)}{r_{10}}\,d\tau_0\right|^2 \leq \int\limits_{r_{10}<2\delta} \frac{d\tau_0}{r_{10}^2}\cdot \int\limits_{r_{10}<2\delta} |\mu(0)|^2\,d\tau_0 \leq 4\pi\cdot 2\delta\|\mu\|^2 = 8\pi\|\mu\|^2\delta;$$

From this it follows that

$$\left|\int\limits_{r_{10}<2\delta} \frac{\mu(0)}{r_{10}}\,d\tau_0\right| \leq \sqrt{8\pi}\,\|\mu\|\,\delta^{1/2}. \tag{100}$$

Since the sphere $r_{10} < 2\delta$ is contained in the sphere $r_{20} < 3\delta$, we find in the same manner that the absolute value of the second integral on the right-hand side of (99) is not greater than $\sqrt{12\pi}\,\|\mu\|\,\delta^{1/2}$. This estimate we denote by (100′).

Since outside the sphere $r_{10} < 2\delta$ the inequality $r_{20} \geqq \frac{r_{10}}{2}$ holds, we find:

$$\left|\frac{1}{r_{10}}-\frac{1}{r_{20}}\right| = \frac{|r_{20}-r_{10}|}{r_{10}r_{20}} \leq \frac{2\delta}{r_{10}^2};$$

applying the BUNYAKOVSKII-SCHWARZ inequality, we obtain:

$$\begin{aligned} \left|\int\limits_{r_{10}\geqq 2\delta} \mu(0)\left[\frac{1}{r_{10}}-\frac{1}{r_{20}}\right]d\tau_0\right| &\leq \sqrt{\int\limits_{r_{10}\geqq 2\delta}|\mu|^2\,d\tau_0}\,\sqrt{\int\limits_{r_{10}\geqq 2\delta}\frac{4\delta^2}{r_{10}^4}\,d\tau_0} \\ &\leq 2\delta\cdot\|\mu\|\sqrt{4\pi\int\limits_{2\delta}^{\infty}\frac{r^2\,dr}{r^4}} = 2\delta\cdot\|\mu\|\sqrt{\frac{4\pi}{2\delta}} = 2\sqrt{2\pi}\cdot\|\mu\|\,\delta^{1/2}. \end{aligned}$$

Inequality (98) follows now from (99), (100), and (100′), wherewith the theorem is proved. We shall make use of this theorem in the sequel.

To conclude this section, we shall present a number of interesting properties of the Newtonian potential with summable density.

We now introduce the definition of generalized derivative due to Academician S. L. SOBOLEV.

Let the summable function $f(x,y,z)$ be defined in a bounded or unbounded region (D). We denote by $\{\varphi\}_l$ the class of functions which are continuous everywhere, have continuous derivatives up to order l, and are such that each

function φ of the class $\{\varphi\}_l$ vanishes outside a certain bounded region (D_φ) which together with its boundary is contained in (D) and which depends on the choice of φ.

Further, let $\omega(x,y,z)$ be a summable function in (D) with the property that for every function φ of the class $\{\varphi\}_l$ the relation

$$\int_{(D)} f \frac{\partial^l \varphi}{\partial x^{l_1} \partial y^{l_2} \partial z^{l_3}} \, d\tau = (-1)^l \int_{(D)} \omega \varphi \, d\tau$$

holds. One says that $\omega(x,y,z)$ is the *generalized derivative* of order l of f and writes:

$$\omega = \frac{\partial^l f}{\partial x^{l_1} \partial y^{l_2} \partial z^{l_3}} \, .$$

We shall show that a Newtonian potential with an everywhere summable density possesses generalized derivatives of first order in every bounded region.

Let (x,y,z) be the coordinates of the point M_0 and (ξ,η,ζ) be those of the point M_1. Let (D) be a bounded region. Then the integral

$$\int_{(D)} \left| \frac{\partial \frac{1}{r_{10}}}{\partial x} \right| d\tau_0$$

is everywhere a bounded and continuous function of M_1, such that the product of $|\mu(1)|$ and this integral is a summable function of M_1. From the FUBINI theorem we conclude that the integral

$$\omega(0) = \int \mu(1) \frac{\partial \frac{1}{r_{10}}}{\partial x} \, d\tau_1$$

converges except for M_0 in a certain set of measure zero. It follows moreover that $\omega(0)$ is summable in an arbitrary bounded region. We shall show that $\omega(0)$ is the generalized derivative of the Newtonian potential.

$$P(0) = \int \mu(1) \frac{1}{r_{10}} \, d\tau_1 \, .$$

For this purpose let $\varphi(x,y,z)$ be a certain function of the class $\{\varphi\}_1$. We then obtain by interchanging the order of integration, the justification for which follows from the FUBINI theorem, that

$$\int\limits_{(D_\varphi)} P(0)\frac{\partial\varphi}{\partial x}d\tau_0=\int\limits_{(D_\varphi)}\frac{\partial\varphi}{\partial x}\left[\int\frac{\mu(1)}{r_{10}}d\tau_1\right]d\tau_0=\int\mu(1)\left[\int\limits_{(D_\varphi)}\frac{\partial\varphi}{\partial x}\frac{1}{r_{10}}d\tau_0\right]d\tau_1$$

$$=\int\mu(1)\left[-\int\limits_{(D_\varphi)}\varphi(0)\frac{\partial\frac{1}{r_{10}}}{\partial x}d\tau_0\right]d\tau_1$$

$$=-\int\limits_{(D_\varphi)}\varphi(0)\left[\int\mu(1)\frac{\partial\frac{1}{r_{10}}}{\partial x}d\tau_1\right]d\tau_0=-\int\limits_{(D_\varphi)}\omega(0)\varphi(0)\,d\tau_0;$$

hence,

$$\int\limits_{(D_\varphi)} P(0)\frac{\partial\varphi}{\partial x}d\tau_0=-\int\limits_{(D_\varphi)}\omega(0)\varphi(0)\,d\tau_0.$$

Since $\varphi(x,y,z)$ is an arbitrary function of the class $\{\varphi\}_1$, this means that $\omega(0)$ is the generalized derivative of $P(0)$. Hence,

$$\omega(0)=\frac{\partial P}{\partial x},$$

as was to be shown.

If μ as well as μ^2 is summable, then one can say still more about $P(0)$. We have already seen that $P(0)$ is then H-continuous. Moreover, the following assertions, which we present without proof, are true:

1. $\frac{\partial P}{\partial x}$ and $\frac{\partial P}{\partial y}$ are square-summable in every bounded region.

2. The generalized derivatives of second order exist. These are everywhere square-summable and are expressed by

$$\frac{\partial^2 P}{\partial x^2}=-\frac{4\pi}{3}\mu_0+\int\mu(1)\frac{\partial^2\frac{1}{r_{10}}}{\partial x^2}d\tau_1,$$

$$\frac{\partial^2 P}{\partial x\,\partial y}=\int\mu(1)\frac{\partial^2\frac{1}{r_{10}}}{\partial x\,\partial y}d\tau_1,\ \ldots.$$

Here the integrals in the usual sense are divergent and are to be interpreted as limits of the functions

$$\int\limits_{r_{10}\gtrsim\varepsilon}\mu(1)\frac{\partial^2\frac{1}{r_{10}}}{\partial x^2}d\tau_1,\qquad \int\limits_{r_{10}\gtrsim\varepsilon}\mu(1)\frac{\partial^2\frac{1}{r_{10}}}{\partial x\,\partial y}d\tau_1,\quad \text{etc.}$$

in L_2 for $\varepsilon\to 0$.

For fixed $\varepsilon>0$ the last integrals are continuous functions of the point M_0. From the formulas for the second derivatives it follows that

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = -4\pi\mu(0),$$

i.e., the Newtonian potential with square-summable density satisfies the POISSON equation wherein the derivatives are to be understood as generalized derivatives.[6]

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[6] The proof of this last assertion may be found in §§17-21 of the book by S. G. MIKHLIN, *The Problem of the Minimum of a Quadratic Functional* (English translation), Holden-Day Inc., San Francisco, 1965.

CHAPTER III

THE NEUMANN PROBLEM AND THE ROBIN PROBLEM

§1. The NEUMANN Problem

With regard to the forms of regions bounded by surfaces (S) we shall distinguish the following cases:

1. In this case, which we shall call the ordinary case, there exists a single surface which bounds a connected region (Fig. 23). Of the two regions bounded by (S) we denote the outer region containing the infinitely distant point by (D_e) and the inner region by (D_i). The normal N to (S) shall always be directed into the region (D_e).

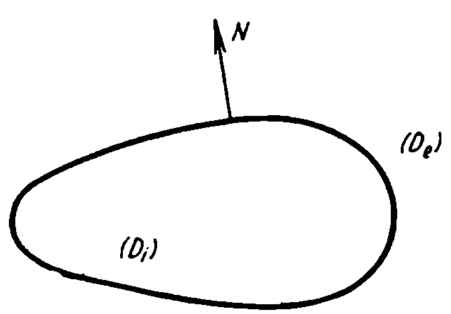

Fig. 23

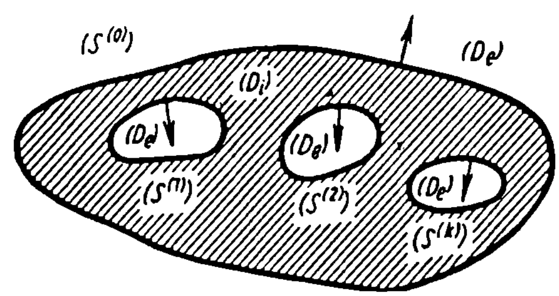

Fig. 24

2. The case (J) in which—as shown in Figure 24—there are several inner boundaries.

We denote by (D_i) the connected region bounded by all the surfaces which does not contain the infinitely distant point, while the rest of space we denote by (D_e). We shall assume that the normal of the boundary surface always points into (D_e). We denote the entire boundary by (S), the outer boundary by $(S^{(0)})$ and the inner boundaries by $(S^{(1)})$, $(S^{(2)})$, . . ., $(S^{(k)})$.

3. The case (E) in which there are several connected regions (Fig. 25). In this case we denote by (D_e) the connected region bounded by all the boundaries which contains the infinitely distant point and by (D_i) the region composing the rest of space.

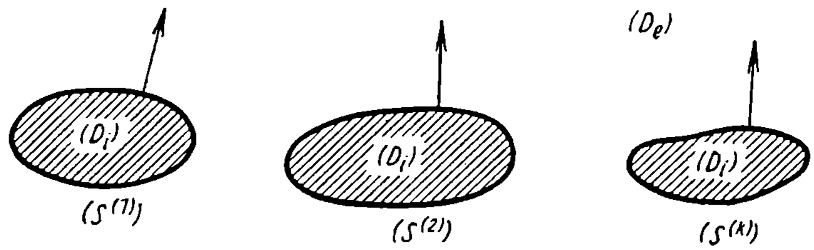

Fig. 25

We shall also here assume that the normals N are always directed into the region (D_e). We let (S) denote the entire boundary, while $(S^{(1)}), (S^{(2)}), \ldots, (S^{(k)})$ denote the boundaries of the individual regions; we always suppose that all boundary surfaces satisfy the LYAPUNOV conditions.

Problem A. *It is desired to find a function V harmonic in the interior of the region (D_i) which satisfies the condition:*

$$\frac{dV_i}{dn} = f \qquad \text{at each point of } (S), \tag{1}$$

where f is a prescribed function continuous on (S).

This is the inner NEUMANN problem. The outer NEUMANN problem is similar:

It is desired to find a function V harmonic in the interior of (D_e) which satisfies the following condition:

$$\frac{dV_e}{dn} = f \qquad \text{at each point of } (S). \tag{2}$$

One easily sees that in certain cases nonzero solutions of the NEUMANN problem exists for $f = 0$. Indeed, in the ordinary case and in case (J) a function U which is constant in (D_i) is harmonic in (D_i) and on the boundary satisfies the condition

$$\frac{dU_i}{dn} = 0. \tag{1'}$$

In case (E) the same condition is met by a function U which assumes arbitrary constant values inside each of the surfaces $(S^{(1)}), \ldots, (S^{(k)})$.

In case (J) a function U, which assumes arbitrary constant values inside each of the surfaces $(S^{(1)}), \ldots, (S^{(k)})$ and is equal to zero outside $(S^{(0)})$, is harmonic in (D_e) and on the boundary satisfies the condition

$$\frac{dU_e}{dn} = 0. \tag{2'}$$

If a function V is harmonic in the interior of (D_i) (or in (D_e) in case (J)) and

is a solution of the NEUMANN problem, then clearly $V + U$ is a solution of the same NEUMANN problem. One can show that this is the extent of the lack of uniqueness in the NEUMANN problem. For this it suffices to prove the following:

1. A function harmonic in the interior of a finite connected region is equal to a constant if its normal derivative vanishes on the boundary of the region.

2. A function harmonic in the interior of an infinite connected region with finite boundary is equal to zero if its normal derivative vanishes.

In order not to interrupt the discussion, we delay the proof of this assertion until the end of the chapter.

Remark. In case (J) one may replace the outer problem by k inner problems and one outer problem. In the same manner, in case (E) the solution of the inner problem may be replaced by the solution of k inner problems.

§2. Replacement of Problem A by Another Problem

In place of Problem A we shall solve the following problem.

Problem B. *It is desired to find the potential of a simple layer on* (S) *which for the inner problem satisfies condition* (1) *and for the outer problem condition* (2).

Let

$$V = \int\limits_{(S_1)} \frac{\mu\, d\sigma_1}{r_{10}} \tag{3}$$

be the potential of a simple layer on (S). In case (J) this integral is represented by the sum

$$\int\limits_{(S_1^{(0)})} \frac{\mu\, d\sigma_1}{r_{10}} + \sum_{l=1}^{k} \int\limits_{(S_1^{(l)})} \frac{\mu\, d\sigma_1}{r_{10}}$$

and in case (E) by the sum

$$\sum_{l=1}^{k} \int\limits_{(S_1^{(l)})} \frac{\mu\, d\sigma_1}{r_{10}} .$$

Remark. We shall denote by $M_0(x,y,z)$ the point for which we calculate V, by $M_1(x_1,y_1,z_1)$ the integration point, and by r_{10} the distance between M_0 and M_1. In place of (S) we shall write (S_1) to indicate that (S) is the location of the point M_1.

As we have seen, the potential of the simple layer with continuous density is a harmonic function in the interior of (D_i) and in the interior of (D_e). Every solution of Problem B is therefore also a solution of Problem A; all other solutions of Problem A differ from that of Problem B by a function U discussed in the preceding section. We shall subsequently see that every solution of

Problem A can be represented as the potential of a simple layer and is hence a solution of Problem B. Problems A and B are therefore equivalent.

We replace Problem B by Problem C, the solution of which leads to the complete solution of Problem B.

Problem C. *It is required to find the potential W of a simple layer which satisfies the following equation:*

$$\frac{dW_i}{dn} - \frac{dW_e}{dn} = -2\zeta\frac{dW}{dn} - 2f \qquad \text{on } (S). \tag{B}$$

The function W which satisfies equation (B) depends on ζ; one easily sees that for $\zeta = 1$ it is the solution of the problem B_i and for $\zeta = -1$ the solution of the problem B_e.

We need only recall the formulas in II, §4:

$$\left.\begin{aligned} \frac{dW_i}{dn} - \frac{dW_e}{dn} &= 4\pi \times \text{density of } W, \\ \frac{dW_i}{dn} + \frac{dW_e}{dn} &= 2\frac{dW}{dn}. \end{aligned}\right\} \tag{4}$$

With the help of these we find for $\zeta = 1$:

$$\frac{dW_i}{dn} - \frac{dW_e}{dn} = -\frac{dW_i}{dn} - \frac{dW_e}{dn} - 2f, \qquad \frac{dW_i}{dn} = -f;$$

so the function W is a solution of the inner Problem B.

Putting $\zeta = -1$, we find:

$$\frac{dW_i}{dn} - \frac{dW_e}{dn} = \frac{dW_i}{dn} + \frac{dW_e}{dn} - 2f, \qquad \frac{dW_e}{dn} = f;$$

from this it follows that in this case W is a solution of the outer Problem B.

We shall denote the density of the potential W by $-\frac{1}{2\pi}\mu$, i.e.,

$$W = -\frac{1}{2\pi}\int\limits_{(S_1)} \frac{\mu\, d\sigma_1}{r_{10}}. \tag{5}$$

The first of equations (4) then assumes the following form:

$$\frac{dW_i}{dn} - \frac{dW_e}{dn} = 4\pi\left(-\frac{1}{2\pi}\mu\right) = -2\mu. \tag{6}$$

Noting that

$$\frac{dW}{dn} = -\frac{1}{2\pi}\int\limits_{(S_1)} \mu\frac{\cos(r_{10}N_0)}{r_{10}^2}\, d\sigma_1,$$

we find from equation (B):

$$-2\mu = \frac{2\zeta}{2\pi}\int\limits_{(S_1)} \mu \frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1 - 2f,$$

$$\mu = -\frac{\zeta}{2\pi}\int\limits_{(S_1)} \mu \frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1 + f.$$

We have thus obtained an integral equation for μ which we shall write in the form

$$\mu(0) = -\frac{\zeta}{2\pi}\int\limits_{(S_1)} \mu(1) \frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1 + f(0) \tag{7}$$

wherein we denote by $F(0)$ a function of M_0 and by $F(1)$ a function of M_1.

The kernel of equation (7) is

$$K(1,0) = -\frac{1}{2\pi}\frac{\cos(r_{10}N_0)}{r_{01}^2}$$

It is unbounded. The equation associated with (7) reads:

$$\vartheta(0) = \zeta\int\limits_{(S_1)} K(0,1)\,\vartheta(1)\,d\sigma_1 + f(0) = -\frac{\zeta}{2\pi}\int\limits_{(S_1)} \vartheta(1)\frac{\cos(r_{01}N_1)}{r_{01}^2}\,d\sigma_1 + f(0).$$

According to agreement, the segment r_{10} is always oriented from M_0 to M_1. We may therefore write the last equation in the form

$$\vartheta(0) = \frac{\zeta}{2\pi}\int\limits_{(S_1)} \vartheta(1)\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1 + f(0) \tag{8}$$

The first term on the right-hand side is here a double-layer potential.

Remark. In the DIRICHLET problem (Ch. IV) equation (8) plays the same role as equation (7) in the NEUMANN problem.

§3. The Formal Solution of Equation (B)

We assume that it is possible to satisfy equation (7)

$$\mu(0) = -\frac{\zeta}{2\pi}\int\limits_{(S_1)} \mu(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1 + f(0)$$

by means of a series

$$\mu(0) = \varrho_0(0) + \zeta\varrho_1(0) + \cdots + \zeta^n\varrho_n(0) + \cdots, \tag{9}$$

which converges uniformly on (S) for sufficiently small values of $|\zeta|$.

If the kernel of equation (7) were bounded, then the existence of such a series would be ensured.

Substituting the series (9) into equation (7) for μ and equating like powers of ζ, we obtain the sequence of equations

$$\left.\begin{aligned}
\varrho_0(0) &= f(0), \\
\varrho_1(0) &= -\frac{1}{2\pi}\int\limits_{(S_1)} \varrho_0(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1, \\
&\ldots\ldots\ldots\ldots\ldots, \\
\varrho_k(0) &= -\frac{1}{2\pi}\int\limits_{(S_1)} \varrho_{k-1}(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1, \\
&\ldots\ldots\ldots\ldots\ldots\ldots.
\end{aligned}\right\} \tag{10}$$

This sequence of equalities makes it possible to determine the functions $\varrho_0, \varrho_1, \ldots$ successively.

Making use of the hypothesis of uniform convergence of series (9), we multiply the series by $-\frac{1}{2\pi}\frac{d\sigma_1}{r_{10}}$ and integrate termwise. We then obtain the following series for the potential W defined by formula (5):

$$W = V_1 + \zeta V_2 + \zeta^2 V_3 + \cdots + \zeta^{k-1} V_k + \cdots \tag{11}$$

with

$$\left.\begin{aligned}
V_1 &= -\frac{1}{2\pi}\int\limits_{(S_1)} \frac{\varrho_0(1)\,d\sigma_1}{r_{10}}, \\
V_2 &= -\frac{1}{2\pi}\int\limits_{(S_1)} \frac{\varrho_1(1)\,d\sigma_1}{r_{10}}, \\
&\ldots\ldots\ldots\ldots, \\
V_k &= -\frac{1}{2\pi}\int\limits_{(S_1)} \frac{\varrho_{k-1}(1)\,d\sigma_1}{r_{10}}, \\
&\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{12}$$

Computing the normal derivatives of the potentials (12), we find that

$$\left.\begin{aligned}
\frac{dV_1}{dn} &= -\frac{1}{2\pi}\int\limits_{(S_1)} \varrho_0(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1, \\
\frac{dV_2}{dn} &= -\frac{1}{2\pi}\int\limits_{(S_1)} \varrho_1(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1, \\
&\ldots\ldots\ldots\ldots\ldots, \\
\frac{dV_k}{dn} &= -\frac{1}{2\pi}\int\limits_{(S_1)} \varrho_{k-1}(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1, \\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{13}$$

Equating now equations (13) and (10), it follows that

$$\frac{dV_1}{dn} = \varrho_1(0), \qquad \frac{dV_2}{dn} = \varrho_2(0), \ldots, \qquad \frac{dV_k}{dn} = \varrho_k(0), \ldots.$$

Replacing $\varrho_0(1)$, $\varrho_1(1)$, ... in (12) by the values found, we obtain:

$$\left.\begin{aligned} V_1 &= -\frac{1}{2\pi}\int\limits_{(S_1)} f(1)\frac{d\sigma_1}{r_{10}}, \\ V_2 &= -\frac{1}{2\pi}\int\limits_{(S_1)} \frac{dV_1}{dn}\frac{d\sigma_1}{r_{10}}, \\ &\ldots\ldots\ldots\ldots, \\ V_k &= -\frac{1}{2\pi}\int\limits_{(S_1)} \frac{dV_{k-1}}{dn}\frac{d\sigma_1}{r_{10}}, \\ &\ldots\ldots\ldots\ldots. \end{aligned}\right\} \tag{12'}$$

These equations make it possible to compute the potentials (12) without making use of the functions (10). The potentials (12′) we shall call the STEKLOV-ROBIN *potentials.*

Equation (11) gives us a formal solution of equation (B). If the series (11) actually converges for $\zeta = 1$ or $\zeta = -1$, then W actually represents a solution of Problem B. The problem is therefore reduced to studying the convergence of the series (11) for $\zeta = 1$ and $\zeta = -1$ with which we shall concern ourselves in the next section.

§4. Investigation of the Iterated Kernels

We have agreed to denote the boundary (S) by (S_1) and its surface element by $d\sigma_1$ if the coordinates of the variable point are x_1, y_1, and z_1; similarly, we will call the same surface (S_2), (S_3), ... if the coordinates of the variable point are (x_2,y_2,z_2), (x_3,y_3,z_3), ...; we shall call the corresponding surface elements $d\sigma_2$, $d\sigma_3$, ..., while we shall denote a function of the point (x_n,y_n,z_n) by $f(n)$.

We now consider the iterated kernels of the kernel $K(1,0) = K_1(1,0)$:

$$K_n(1,0) = \int\limits_{(S_2)} K(1,2)\,K_{n-1}(2,0)\,d\sigma_2.$$

We first study the kernel

$$K_2(1,0) = \frac{1}{4\pi^2}\int\limits_{(S_2)} \frac{\cos(r_{12}N_2)}{r_{12}^2}\,\frac{\cos(r_{20}N_0)}{r_{20}^2}\,d\sigma_2.$$

This kernel, considered as a function of the point M_1, is the double-layer potential with density $-\dfrac{\cos(r_{20}N_0)}{4\pi^2 r_{20}^2}$; this density is integrable and continuous

everywhere except at the point M_0 in a neighborhood of which it is possibly unbounded. From a remark in II, §3 it now follows that $K_2(1,0)$ is continuous as a function of M_1 as long as M_1 does not coincide with M_0; $K_2(1,0)$ may be unbounded in a neighborhood of M_0. Considering $K_2(1,0)$ as the normal derivative of the simple-layer potential, we conclude similarly that $K_2(1,0)$ is continuous as a function of M_0 if M_0 does not coincide with M_1. We can make the same statements about the iterated kernels following $K_2(1,0)$. We shall show that one of the iterated kernels is bounded, so that the next following kernel is certainly continuous. In the following considerations we shall invoke the boundedness of the functions and make no special mention of their continuity.

We assume that (S) is a LYAPUNOV surface, so that

$$(N_1 N_0) < E r_{10}^{\lambda}.$$

In this case we can show that *the iterated kernel* $K_n(1,0)$ *belonging to the kernel*

$$K(1,0) = -\frac{1}{2\pi}\frac{\cos(r_{10}N_0)}{r_{10}^2}$$

is bounded if its order n satisfies the inequalities

$$2-(n-1)\lambda \geq 0, \quad 2-n\lambda < 0$$

For this purpose we first of all recall the inequality obtained in I, §1:

$$|\cos(r_{10}N_0)| < a r_{10}^{\lambda}.$$

With the help of this inequality we find:

$$\left|\frac{1}{2\pi}\cdot\frac{\cos(r_{10}N_0)}{r_{10}^2}\right| r_{10}^{2-\lambda} < \frac{a}{2\pi} = B.$$

Putting

$$K(1,0) = \frac{C(1,0)}{r_{10}^{2-\lambda}},$$

it follows that $C(1,0)$ is a continuous and bounded function.

We now wish to estimate the second iterated kernel. We have:

$$K_2(1,0) = \int\limits_{(S_2)} K(1,2)\,K(2,0)\,d\sigma_2 = \int\limits_{(S_2)} \frac{C(1,2)\,C(2,0)}{r_{12}^{2-\lambda}\,r_{20}^{2-\lambda}}\,d\sigma_2.$$

We consider the LYAPUNOV sphere and the sphere of radius $2r_{10}$ about the point M_0 (Fig. 26).

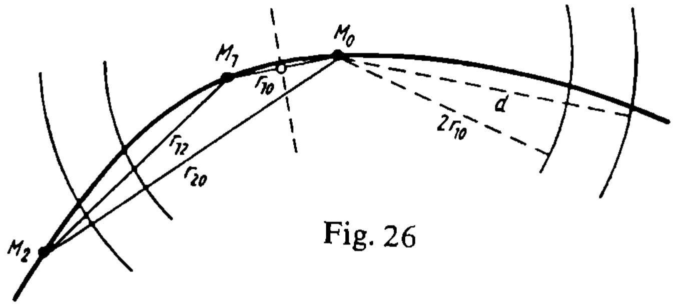

Fig. 26

Let (Σ_2) and (σ_2) be the subregions of (S) cut out by these spheres. Then

$$\begin{aligned} K_2(1,0) = &\int\limits_{(S_2-\Sigma_2)} \frac{C(1,2)\,C(2,0)}{r_{12}^{2-\lambda}\,r_{20}^{2-\lambda}}\,d\sigma_2 \\ &+ \int\limits_{(\Sigma_2-\sigma_2)} \frac{C(1,2)\,C(2,0)}{r_{12}^{2-\lambda}\,r_{20}^{2-\lambda}}\,d\sigma_2 + \int\limits_{(\sigma_2)} \frac{C(1,2)\,C(2,0)}{r_{12}^{2-\lambda}\,r_{20}^{2-\lambda}}\,d\sigma_2 . \end{aligned} \tag{14}$$

The first of these three integrals is bounded, since r_{12} and r_{20} are greater than $\frac{d}{2}$ in the region of integration if r_{10} is less than $\frac{d}{2}$.

We consider the second integral. Since the point M_2 lies outside the sphere of radius $2r_{10}$,

$$r_{12} > \frac{1}{2}\,r_{20}\,.$$

Introducing cylindrical coordinates with M_0 as pole and the tangent plane at M_0 as polar plane, we obtain:

$$\left|\int\limits_{(\Sigma_2-\sigma_2)} \frac{C(1,2)\,C(2,0)}{r_{12}^{2-\lambda}\,r_{20}^{2-\lambda}}\,d\sigma_2\right| < B^2\cdot 2^{2-\lambda}\int\limits_{(\Sigma_2-\sigma_2)} \frac{d\sigma_2}{r_{20}^{4-2\lambda}}$$

$$< B^2\cdot 4\pi\cdot 2^{2-\lambda}\int\limits_{r_{10}}^{d} \frac{\varrho\,d\varrho}{\varrho^{4-2\lambda}} = \frac{B^2\cdot 2^{2-\lambda}\cdot 4\pi}{2-2\lambda}\left\{\frac{1}{r_{10}^{2-2\lambda}} - \frac{1}{d^{2-2\lambda}}\right\}$$

We hence see that the product of the second integral and $r_{10}^{2-2\lambda}$ is bounded.

To estimate that last integral in (14), we construct the normal plane to M_0M_1 through the midpoint of r_{10}. This plane decomposes (σ_2) into two parts, namely the subregion (σ_0) containing the point M_0 and the subregion (σ_1) containing the point M_1.

The last integral in (14) is then the sum of the two integrals over (σ_0) and

(σ_1); we shall now consider the first of these. If one integrates over (σ_0), then r_{12} is greater than $\frac{1}{2}r_{10}$. From this it follows that

$$\left|\int\limits_{(\sigma_0)} \frac{C(1,2)\,C(2,0)}{r_{12}^{2-\lambda} r_{20}^{2-\lambda}}\, d\sigma_2\right| < \frac{B^2 \cdot 2^{2-\lambda}}{r_{10}^{2-\lambda}} \int\limits_{(\sigma_0)} \frac{d\sigma_2}{r_{20}^{2-\lambda}}$$

$$< \frac{B^2 \cdot 2^{2-\lambda} \cdot 2^2 \pi}{r_{10}^{2-\lambda}} \int\limits_0^{2r_{10}} \frac{\varrho\, d\varrho}{\varrho^{2-\lambda}} = \frac{B^2 \cdot 2^{2-\lambda} \cdot 4\pi}{r_{10}^{2-\lambda}\,\lambda} (2\, r_{10})^{\lambda}.$$

Hence the product of this integral and the power $r_{10}^{2-2\lambda}$ is bounded.

We denote by (σ_2') the subregion of (S) cut out by the sphere of radius $3r_{10}$ about M_1 which therefore contains (σ_2) in its interior; if we take account of the fact that in integrating over (σ_1) r_{20} is greater than $\frac{1}{2}r_{10}$, then we obtain as above:

$$\left|\int\limits_{(\sigma_1)} \frac{C(1,2)\,C(2,0)}{r_{12}^{2-\lambda} r_{20}^{2-\lambda}}\, d\sigma_2\right| < \frac{B^2 \cdot 2^{2-\lambda}}{r_{10}^{2-\lambda}} \int\limits_{(\sigma_1)} \frac{d\sigma_2}{r_{12}^{2-\lambda}}$$

$$< \frac{B^2 \cdot 2^{2-\lambda}}{r_{10}^{2-\lambda}} \int\limits_{(\sigma_2')} \frac{d\sigma_2}{r_{12}^{2-\lambda}} < \frac{B^2 \cdot 2^{2-\lambda} \cdot 4\pi}{r_{10}^{2-\lambda}\,\lambda} (3\, r_{10})^{\lambda}.$$

Gathering together the results obtained, we find:

$$\left|K_2(1,0)\, r_{10}^{2-2\lambda}\right| < B_1, \qquad K_2(1,0) = \frac{C_2(1,0)}{r_{10}^{2-2\lambda}};$$

B_1 is here a constant, and $C_2(1,0)$ is a bounded function.

Repeating the same considerations for the kernel

$$K_3(1,0) = \int\limits_{(S_2)} K(1,2)\, K_2(2,0)\, d\sigma_2$$

we obtain:

$$K_3(1,0) = \frac{C_3(1,0)}{r_{10}^{2-3\lambda}},$$

where $C_3(1,0)$ is a bounded function.

Proceeding in this manner, we finally find that

$$K_{n-1}(1,0) = \frac{C_{n-1}(1,0)}{r_{10}^{2-(n-1)\lambda}}, \quad \text{if } 2-(n-1)\lambda \neq 0, \text{ and}$$

$$K_{n-1}(1,0) = C_{n-1}(1,0)\left|\ln r_{10}\right|, \quad \text{if } 2-(n-1)\lambda = 0.$$

The boundedness of $K_n(1,0)$ is herewith established.

§5. The Actual Solution of Equation (B)

From the theory of integral equations the solution of equation (7)

$$\mu(0) = \frac{-\zeta}{2\pi} \int\limits_{(S_1)} \mu(1) \frac{\cos(r_{10} N_0)}{r_{10}^2} d\sigma_1 + f(0)$$
$$= \zeta \int\limits_{(S_1)} K(1, 0)\, \mu(1)\, d\sigma_1 + f(0)$$

also satisfies the equation

$$\mu(0) = \zeta^n \int\limits_{(S_1)} K_n(1, 0)\, \mu(1)\, d\sigma_1 + \Sigma_n(0); \tag{15}$$

where $\Sigma_n(0)$ is the sum of the first n terms in the expansion of μ in powers of ζ:

$$\Sigma_n(0) = \varrho_0(0) + \zeta \varrho_1(0) + \cdots + \zeta^{n-1} \varrho_{n-1}(0)$$
$$= f(0) + \zeta \int\limits_{(S_1)} K(1, 0) f(1)\, d\sigma_1 + \cdots + \zeta^{n-1} \int\limits_{(S_1)} K_{n-1}(1, 0) f(1)\, d\sigma_1 .$$

If the kernel $k_n(1,0)$ is bounded, then one may introduce the corresponding FREDHOLM determinant $D(\zeta)$, its first subdeterminant $D(\zeta,1,0)$, and the resolvent

$$\frac{D(\zeta, 1, 0)}{D(\zeta)} \tag{16}$$

The expressions $D(\zeta)$ and $D(\zeta,1,0)$ can be expanded in series of powers of ζ which converge for arbitrary ζ. Since (16) is the resolvent for equation (15),

$$\mu(0) = \Sigma_n(0) + \zeta^n \int\limits_{(S_1)} \Sigma_n(1) \frac{D_1(\zeta, 1, 0)}{D(\zeta)} d\sigma_1 = \frac{D_1(\zeta, 0)}{D_1(\zeta)} . \tag{17}$$

The function $\mu(0)$ can therefore be represented as a fraction, the numerator and denominator of which can be expanded in series of powers of ζ which converge for all values of ζ. The coefficients of the series for $D_1(\zeta,0)$ are continuous functions of M_0, and this series converges uniformly on (S) for arbitrary values of ζ. We denote the denominator of the fraction in the last equation by $D_1(\zeta)$ instead of $D(\zeta)$, while we hereby assume that all common factors in this fraction have been cancelled. If $\zeta_1 \neq 0$ is the zero of $D_1(\zeta)$ of smallest absolute value, then the series (9) converges uniformly on (S) for every fixed ζ of the disc $|\zeta| < |\zeta_1|$. The potential

$$W(0) = -\frac{1}{2\pi} \int\limits_{(S_1)} \frac{D_1(\zeta, 1)}{D_1(\zeta)} \frac{d\sigma_1}{r_{10}}$$

is the solution of equation (B). If $\zeta = 1$ and $\zeta = -1$ are not zeros of the function $D_1(\zeta)$, then equation (B) has solutions for these values.

Hence, the solution of the NEUMANN problem reduces to the question of the conditions under which $\zeta = 1$ and $\zeta = -1$ are not poles of the mermorphic function $\mu(0)$.

In investigating this question we first of all prove the following four theorems.

1. *In the ordinary case and in case* (E) $\zeta = -1$ *is not a pole of the function* $\mu(0)$. *This assertion does not in general carry over to the case* (J).

In all cases we have:

2. *The poles of the function* $\mu(0)$ *are real.*
3. *The function* $\mu(0)$ *has no pole between* -1 *and* $+1$.
4. *All poles of the function* $\mu(0)$ *are simple.*

§6. A Lemma

To prove the theorems formulated in §5 we must make use of GREEN'S identities in which U and V are simple-layer potentials:

$$\int\limits_{(D_i)}\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\frac{\partial V}{\partial y}+\frac{\partial U}{\partial z}\frac{\partial V}{\partial z}\right)d\tau=\int\limits_{(S)}U\frac{dV_i}{dn}\,d\sigma=\int\limits_{(S)}V\frac{dU_i}{dn}\,d\sigma, \quad (18)$$

$$\int\limits_{(D_i)}\left[\left(\frac{\partial V}{\partial x}\right)^2+\left(\frac{\partial V}{\partial y}\right)^2+\left(\frac{\partial V}{\partial z}\right)^2\right]d\tau=\int\limits_{(S)}V\frac{dV_i}{dn}\,d\sigma, \quad (19)$$

$$\int\limits_{(S)}\left(U\frac{dV_i}{dn}-V\frac{dU_i}{dn}\right)d\sigma=0. \quad (20)$$

Analogous formulas hold for the region (D_e). We proved these identities for functions which are harmonic in (D_i). The potential of a simple layer is in general not harmonic in (D_i), but rather in the interior of (D_i). Only in the case of an H-continuous density μ can one assert from the considerations in II, §7 that the first derivatives of such a potential are H-continuous in (D_i) so that the GREEN identities may be applied.

In this section we shall prove *the validity of formula* (*18*) *for potentials of simple layers with continuous density* which will suffice for the following considerations. Formulas (19) and (20) are consequences of (18).

Let the points M_1 and M_3 be in (D_e). Then $\frac{1}{r_{10}}$ and $\frac{1}{r_{30}}$ are harmonic functions of the point M_0 in (D_i) which have continuous derivatives of arbitrary order and to which the GREEN identity is applicable:

$$\int_{(D_i)} \left(\frac{\partial \frac{1}{r_{10}}}{\partial x} \cdot \frac{\partial \frac{1}{r_{30}}}{\partial x} + \frac{\partial \frac{1}{r_{10}}}{\partial y} \cdot \frac{\partial \frac{1}{r_{30}}}{\partial y} + \frac{\partial \frac{1}{r_{10}}}{\partial z} \cdot \frac{\partial \frac{1}{r_{30}}}{\partial z} \right) d\tau$$
$$= \int_{(S)} \frac{1}{r_{10}} \frac{d \frac{1}{r_{30}}}{d n} \, d\sigma = - \int_{(S)} \frac{1}{r_{10}} \frac{\cos(r_{03} N_0)}{r_{03}^2} \, d\sigma . \tag{21}$$

If one holds the point M_3 in (D_e) fixed, then both sides of equation (21) are everywhere continuous functions of M_1, being respectively the integral over (D_i) of a sum of first derivatives of Newtonian potentials with continuous density and the integral over (S) of a potential of a simple layer with continuous density. Formula (21) therefore also holds when the point M_1 is on (S). Let now M_1 be a point of (S). From a remark in II, §13 the left-hand side of (21), as a function of M_3, is continuous everywhere with exception of the point M_1. If M_3 coincides with a point M_2 of (S) distinct from M_1, then from the Addendum in II, §3 the quantity $\frac{2\pi}{r_{12}}$ is to be added to the right-hand side.

We thus arrive at the identity

$$\int_{(D_i)} \left(\frac{\partial \frac{1}{r_{10}}}{\partial x} \cdot \frac{\partial \frac{1}{r_{20}}}{\partial x} + \cdots \right) d\tau = \frac{2\pi}{r_{12}} - \int_{(S)} \frac{1}{r_{10}} \frac{\cos(r_{02} N_0)}{r_{02}^2} \, d\sigma . \tag{22}$$

Let μ and ν be the densities of the simple-layer potentials U and V. Then from (22)

$$\int_{(S_2)} \mu(2) \left\{ \int_{(S_1)} \nu(1) \left[\int_{(D_i)} \left(\frac{\partial \frac{1}{r_{10}}}{\partial x} \cdot \frac{\partial \frac{1}{r_{20}}}{\partial x} + \cdots \right) d\tau \right] d\sigma_1 \right\} d\sigma_2$$
$$= \int_{(S_2)} \mu(2) \left\{ \int_{(S_1)} \nu(1) \left[\frac{2\pi}{r_{12}} - \int_{(S)} \frac{1}{r_{10}} \frac{\cos(r_{\overline{02}} N_0)}{r_{02}^2} \, d\sigma \right] d\sigma_1 \right\} d\sigma_2 . \tag{23}$$

If we interchange the order of integration on the left-hand side of (23), the justification for which we shall give in a moment, we obtain the formula

$$\int_{(S_2)} \mu(2) \left\{ \int_{(S_1)} \nu(1) \left[\int_{(D_i)} \frac{\partial \frac{1}{r_{10}}}{\partial x} \cdot \frac{\partial \frac{1}{r_{20}}}{\partial x} \, d\tau \right] d\sigma_1 \right\} d\sigma_2$$
$$= \int_{(D_i)} \left[\int_{(S_1)} \nu(1) \frac{\partial \frac{1}{r_{10}}}{\partial x} \, d\sigma_1 \right] \cdot \left[\int_{(S_2)} \mu(2) \frac{\partial \frac{1}{r_{20}}}{\partial x} \, d\sigma_2 \right] d\tau = \int_{(D_i)} \frac{\partial U}{\partial x} \cdot \frac{\partial V}{\partial x} \, d\tau$$

and formulas analogous to this.

Hence, the left-hand side of (23) assumes the form

$$\int\limits_{(D_i')}\left(\frac{\partial U}{\partial x}\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\frac{\partial V}{\partial y}+\frac{\partial U}{\partial z}\frac{\partial V}{\partial z}\right)d\tau\ ,$$

i.e., it becomes the left-hand side of (18).

The right-hand side of formula (23) upon interchange of the order of integration, the justification for which will also be given shortly, gains the appearance:

$$\begin{aligned}
2\pi\int\limits_{(S_2)}\mu(2)&\left(\int\limits_{(S_1)}\nu(1)\frac{1}{r_{12}}d\sigma_1\right)d\sigma_2\\
&-\int\limits_{(S)}\left[\int\limits_{(S_1)}\nu(1)\frac{1}{r_{10}}d\sigma_1\right]\cdot\left[\int\limits_{(S_2)}\mu(2)\frac{\cos(r_{02}N_0)}{r_{02}^2}d\sigma_2\right]d\sigma\\
&=2\pi\int\limits_{(S)}\mu(0)V(0)\,d\sigma-\int\limits_{(S)}V(0)\left[\int\limits_{(S_2)}\mu(2)\frac{\cos(r_{02}N_0)}{r_{02}^2}d\sigma_2\right]d\sigma\\
&=\int\limits_{(S)}V(0)\left\{2\pi\mu(0)-\int\limits_{(S_2)}\mu(2)\frac{\cos(r_{02}N_0)}{r_{02}^2}d\sigma_2\right\}d\sigma\\
&=\int\limits_{(S)}V(0)\frac{dU_i}{dn_0}d\sigma\ ,
\end{aligned}$$

for one has

$$\begin{aligned}
2\pi\mu(0)-\int\limits_{(S_2)}\mu(2)\frac{\cos(r_{02}N_0)}{r_{02}^2}d\sigma_2&=2\pi\mu(0)+\int\limits_{(S_2)}\mu(2)\frac{\cos(r_{20}N_0)}{r_{20}^2}d\sigma_2\\
&=2\pi\mu(0)+\frac{dU}{dn_0}=\frac{dU_i}{dn_0}\ .
\end{aligned}$$

The right-hand side of (23) is therefore equal to the right-hand side of (18). Assuming the interchange of the order of integration to have been justified, this then completes the proof of formula (18).

From I, §6 to justify the interchange of the order of integration on the left-hand side of (23), it suffices to prove the existence of the integral

$$\int\limits_{(S_2)}|\mu(2)|\left\{\int\limits_{(S_1)}|\nu(1)|\left[\int\limits_{(D_i')}\left|\frac{\partial\frac{1}{r_{10}}}{\partial x}\right|\cdot\left|\frac{\partial\frac{1}{r_{20}}}{\partial x}\right|d\tau\right]d\sigma_1\right\}d\sigma_2. \tag{24}$$

First of all,

$$\left|\frac{\partial\frac{1}{r_{10}}}{\partial x}\right|\le\frac{1}{r_{10}^2}\ ,\qquad\left|\frac{\partial\frac{1}{r_{20}}}{\partial x}\right|\le\frac{1}{r_{20}^2}\ ,$$

and hence

$$\int\limits_{(D_i)} \left|\frac{\partial \frac{1}{r_{10}}}{\partial x}\right| \cdot \left|\frac{\partial \frac{1}{r_{20}}}{\partial x}\right| d\tau < \int\limits_{(D_i)} \frac{1}{r_{10}^2} \cdot \frac{1}{r_{20}^2}\, d\tau .$$

We shall establish the validity of the inequality

$$\int\limits_{(D_i)} \frac{1}{r_{10}^2} \cdot \frac{1}{r_{20}^2}\, d\tau < \frac{C}{r_{12}} . \tag{25}$$

For this purpose we consider the sphere of radius $2r_{12}$ about the point M_1 and denote the portion of the sphere lying in (D_i) by (ω). The plane normal to M_1M_2 passing through the midpoint of this segment decomposes (ω) into two subregions (ω') and (ω''); let (ω') contain the point M_1 and (ω'') the point M_2.

The integral (25) may be written as the sum of the corresponding integrals over $(D_i - \omega)$, (ω'), and (ω''). We shall estimate each of these integrals.

In the region (ω') $r_{20} > \frac{r_{12}}{2}$ and hence

$$\int\limits_{(\omega')} \frac{1}{r_{10}^2} \cdot \frac{1}{r_{20}^2}\, d\tau < \frac{4}{r_{12}^2} \int\limits_{(\omega')} \frac{d\tau}{r_{10}^2} < \frac{4}{r_{12}^2} \int\limits_{(\omega)} \frac{d\tau}{r_{10}^2} < \frac{4}{r_{12}^2} \cdot 4\pi\,(2\,r_{12}) = \frac{32\pi}{r_{12}} \cdot$$

We have here made use of the fact that the integral over (ω') of a positive function is less than the integral over (ω), while this in turn is less than the integral over the entire sphere of radius $2r_{12}$; the value of the last integral was established in II, §11.

Since (ω'') is contained in the sphere of radius $3r_{12}$ about M_2, we obtain an analogous estimate for the integral over (ω'').

In the region $(D_i - \omega)$ $r_{10} > 2r_{12}$ and hence $\frac{r_{12}}{r_{10}} < \frac{1}{2}$. Moreover, $|r_{10} - r_{20}| \leqq r_{12}$, and hence

$$\left|1 - \frac{r_{20}}{r_{10}}\right| < \frac{1}{2} \cdot$$

So

$$\frac{1}{2} < \frac{r_{20}}{r_{10}}$$

and hence

$$\frac{1}{r_{20}} < \frac{2}{r_{10}} \cdot$$

Let R be the diameter of the region (D_i). The integral of a positive function over $(D_i - \omega)$ is then less than the integral over the spherical shell with radii $2r_{12}$ and R and midpoint M_1. We therefore obtain:

$$\int\limits_{(D_i-\omega)} \frac{1}{r_{10}^2} \frac{1}{r_{20}^2}\, d\tau < 4 \int\limits_{(D_i-\omega)} \frac{d\tau}{r_{10}^4} < 4 \cdot 4\pi \int\limits_{2r_{12}}^{R} \frac{\varrho^2}{\varrho^4}\, d\varrho = 16\pi\left(\frac{1}{2r_{12}} - \frac{1}{R}\right) < \frac{8\pi}{r_{12}} \cdot$$

Therefore all the integrals can be estimated by a quantity of the form $\frac{A}{r_{12}}$, wherewith the estimate (25) is proved. From this it follows that

$$\int\limits_{(S_1)} |\nu(1)| \left[\int\limits_{(D_i)} \left| \frac{\partial \frac{1}{r_{10}}}{\partial x} \right| \cdot \left| \frac{\partial \frac{1}{r_{20}}}{\partial x} \right| d\tau \right] d\sigma_1$$

is an everywhere bounded function of M_2, and the integral over (S_2) certainly exists. This completes the proof of the justification for interchanging the order of integration on the left-hand side of (23).

Repeating the considerations of §4, we arrive at the estimate

$$\int\limits_{(S)} \frac{1}{r_{10}} \cdot \frac{|\cos(r_{20} N_0)|}{r_{20}^2} d\sigma < \frac{C}{r_{12}^{1-\lambda}};$$

hence, the integral

$$\int\limits_{(S_2)} |\mu(2)| \left\{ \int\limits_{(S_1)} |\nu(1)| \left[\int\limits_{(S)} \frac{1}{r_{10}} \frac{|\cos(r_{20} N_0)|}{r_{20}^2} d\sigma \right] d\sigma_1 \right\} d\sigma_2$$

exists; this shows that the interchange of the order of integration on the right-hand side of (23) is also justified. This now completes the proof of formula (18).

In addition to the identities (18) and (19), we also prove the following assertions of which we shall later make use.

1. If for a potential of a simple layer the equation $\frac{dV_e}{dn} = 0$ holds for all points of (S), then the potential V and its density in the ordinary case and in case (E) is everywhere equal to zero; in case (J) V is equal to zero outside of $(S^{(0)})$ and is constant inside $(S^{(1)}), (S^{(2)}), \ldots, (S^{(k)})$.

Indeed, $\frac{dV_e}{dn} = 0$ implies

$$\int\limits_{(D_e)} \left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 \right] d\tau = -\int\limits_{(S)} V \frac{dV_e}{dn} d\sigma = 0, \qquad (19')$$

i.e., in (D_e)

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} = \frac{\partial V}{\partial z} = 0, \qquad V = \text{const.}$$

At infinity however $V = 0$; so in the connected region containing the infinitely distant point V is equal to zero. This connected region in the ordinary case and in case (E) coincides with (D_e). In case (J) this is not the case, since the connected region does not here contain the regions interior to $(S^{(1)}), (S^{(2)}), \ldots, (S^{(k)})$ in which V is constant.

Since in the ordinary case and in case (E) the potential V is equal to zero in

(D_e), it also vanishes on the entire boundary (S). Formula (19) shows that V is constant in (D_i), but since it vanishes on (S) it is also equal to zero in (D_i).

If V is everywhere zero, then

$$0 = \frac{dV_i}{dn} - \frac{dV_e}{dn} = 4\pi\mu, \qquad \mu = 0.$$

2. If the potential V is zero in (D_i), then it is everywhere zero. For since V vanishes on (S), it follows from formula (19′) that the derivatives of V in (D_e) are equal to zero and hence V is constant in (D_e). Since V vanishes on (S), V is also zero in (D_e).

Addendum. Let U be harmonic in (D_i) and let V be the potential of a simple layer. One easily sees that formula (18), and hence formulas (19) and (20), are valid.

If M_1 lies in (D_e), then $\frac{1}{r_{10}}$ is a harmonic function of the point M_0 in (D_i), and hence

$$\begin{aligned}\int\limits_{(D_i)}\left(\frac{\partial U}{\partial x}\frac{\partial\frac{1}{r_{10}}}{\partial x} + \frac{\partial U}{\partial y}\frac{\partial\frac{1}{r_{10}}}{\partial y} + \frac{\partial U}{\partial z}\frac{\partial\frac{1}{r_{10}}}{\partial z}\right)d\tau \\ = \int\limits_{(S)}\frac{dU_i}{dn}\frac{1}{r_{10}}\,d\sigma = -\int\limits_{(S)}U\frac{\cos(r_{01}N_0)}{r_{01}^2}\,d\sigma.\end{aligned} \tag{26}$$

Letting the point M_1 in (26) tend toward the surface (S), multiplying the equation so obtained by $\mu(1)$, integrating over (S), and interchanging the order of integration, we arrive at the desired formula (18).

As a consequence of formula (20) we obtain:

$$V(0) = \frac{1}{4\pi}\int\limits_{(S)}\frac{dV_i}{dn}\frac{d\sigma}{r_{10}} + \frac{1}{4\pi}\int\limits_{(S)}V_i\frac{\cos(r_{10}N)}{r_{10}^2}\,d\sigma, \quad \text{for } M_0 \text{ in } (D_i),$$

$$0 = \frac{1}{4\pi}\int\limits_{(S)}\frac{dV_i}{dn}\frac{d\sigma}{r_{10}} + \frac{1}{4\pi}\int\limits_{(S)}V_i\frac{\cos(r_{10}N)}{r_{10}^2}\,d\sigma, \quad \text{for } M_0 \text{ in } (D_e);$$

analogous formulas are obtained for V_e and $\frac{dV_e}{dn}$. The derivation of these relations in I, §8 was also based only on formula (20).

§7. Proof of the Theorems in §5

We now proceed to the proof of the theorems in §5 and suppose that the function $\mu(0)$ has a pole of kth order at ζ_0. We put then

$$\mu(0) = \frac{D_1(\zeta, 0)}{D_1(\zeta)} = \frac{\vartheta_0(0)}{(\zeta-\zeta_0)^k} + \frac{\vartheta_1(0)}{(\zeta-\zeta_0)^{k-1}} + \cdots, \tag{17'}$$

where $\vartheta_0(0)$ is not identically zero. Then

$$W(0) = -\frac{1}{2\pi}\int_{(S_1)} \frac{\mu(1)\, d\sigma_1}{r_{10}} = \frac{1}{(\zeta-\zeta_0)^k} W^{(0)} + \frac{1}{(\zeta-\zeta_0)^{k-1}} W^{(1)} + \cdots \tag{27}$$

with

$$W^{(0)} = -\frac{1}{2\pi}\int_{(S_1)} \frac{\vartheta_0(1)\, d\sigma_1}{r_{10}},$$

$$W^{(1)} = -\frac{1}{2\pi}\int_{(S_1)} \frac{\vartheta_1(1)\, d\sigma_1}{r_{10}},$$

$$\cdots\cdots\cdots\cdots\cdots;$$

$W^{(0)}$ is here not identically zero, for then $\vartheta_0(0) \equiv 0$.

1. We substitute W into the equation

$$\frac{dW_i}{dn} - \frac{dW_e}{dn} = -\zeta\left(\frac{dW_i}{dn} + \frac{dW_e}{dn}\right) - 2f \tag{B}$$

$$= -\zeta_0\left(\frac{dW_i}{dn} + \frac{dW_e}{dn}\right) - 2f - (\zeta-\zeta_0)\left(\frac{dW_i}{dn} + \frac{dW_e}{dn}\right)$$

and compare coefficients of $\dfrac{1}{(\zeta-\zeta_0)^k}$; we thus obtain:

$$\frac{dW_i^{(0)}}{dn} - \frac{dW_e^{(0)}}{dn} = -\zeta_0\left(\frac{dW_i^{(0)}}{dn} + \frac{dW_e^{(0)}}{dn}\right). \tag{28}$$

One then arrives at the equation

$$(1+\zeta_0)\frac{dW_i^{(0)}}{dn} = (1-\zeta_0)\frac{dW_e^{(0)}}{dn}. \tag{29}$$

Multiplying equation (29) by W and integrating over (S), we find with the help of (19) and (19′):

$$(1+\zeta_0)\int_{(D_i)}\sum\left(\frac{\partial W^{(0)}}{\partial x}\right)^2 d\tau = -(1-\zeta_0)\int_{(D_e)}\sum\left(\frac{\partial W^{(0)}}{\partial x}\right)^2 d\tau. \tag{30}$$

For $\zeta_0 = -1$ it now follows that

$$\int_{(D_e)} \sum \left(\frac{\partial W^{(0)}}{\partial x}\right)^2 d\tau = 0.$$

Hence, $W^{(0)}$ is constant in (D_e) for this value of ζ. In the ordinary case and in case (E) $W^{(0)}$ is then everywhere equal to zero, whence it follows that $\vartheta_0(0)$ is everywhere equal to zero. This contradicts the initial hypothesis. Hence, neither in the ordinary case nor in the case (E) can $\zeta_0 = -1$ be a pole of the function μ.

2. We now suppose that ζ_0 is a complex number. Then $\vartheta_0(0)$ and $W^{(0)}$ may also be complex. We put

$$W^{(0)} = V^{(1)} + i\, V^{(2)};$$

Equation (29) then reads:

$$(1+\zeta_0)\left(\frac{d V_i^{(1)}}{d n} + i\,\frac{d V_i^{(2)}}{d n}\right) = (1-\zeta_0)\left(\frac{d V_e^{(1)}}{d n} + i\,\frac{d V_e^{(2)}}{d n}\right). \tag{31}$$

Multiplying both sides of (31) by $V^{(1)} - iV^{(2)}$ gives:

$$\begin{aligned} (1+\zeta_0)&\left[V^{(1)}\frac{d V_i^{(1)}}{d n} + V^{(2)}\frac{d V_i^{(2)}}{d n} + i\left(V^{(1)}\frac{d V_i^{(2)}}{d n} - V^{(2)}\frac{d V_i^{(1)}}{d n}\right)\right] \\ = (1-\zeta_0)&\left[V^{(1)}\frac{d V_e^{(1)}}{d n} + V^{(2)}\frac{d V_e^{(2)}}{d n} + i\left(V^{(1)}\frac{d V_e^{(2)}}{d n} - V^{(2)}\frac{d V_e^{(1)}}{d n}\right)\right]. \end{aligned} \tag{32}$$

If this equation is integrated over (S), then according to (20) the integrals of the quantities in parentheses is zero, and we obtain:

$$\begin{aligned} (1+\zeta_0)&\int_{(D_i)} \sum\left[\left(\frac{\partial V^{(1)}}{\partial x}\right)^2 + \left(\frac{\partial V^{(2)}}{\partial x}\right)^2\right] d\tau \\ &= -(1-\zeta_0)\int_{(D_e)} \sum\left[\left(\frac{\partial V^{(1)}}{\partial x}\right)^2 + \left(\frac{\partial V^{(2)}}{\partial x}\right)^2\right] d\tau. \end{aligned} \tag{33}$$

Denoting the two integrals occurring in (33) by J_i and J_e and putting $\zeta_0 = \xi_0 + i\eta_0$, we further obtain:

$$(1+\xi_0)J_i + (1-\xi_0)J_e + i\eta_0(J_i - J_e) = 0. \tag{33'}$$

Now J_i and J_e cannot both vanish, for then $V^{(1)} = V^{(2)} \equiv 0$, so that $W^{(0)} \equiv 0$ and $\vartheta_0(0) \equiv 0$, contrary to hypothesis. On the other hand, from (33)′ there follows the homogeneous linear system for J_i and J_e

$$\begin{aligned} (1+\xi_0)J_i + (1-\xi_0)J_e &= 0 \\ \eta_0\, J_i - \quad \eta_0\, J_e &= 0. \end{aligned}$$

The determinant of the coefficient of this system must vanish:

$$-(1+\xi_0)\eta_0 - (1-\xi_0)\eta_0 = -2\eta_0 = 0.$$

From this it follows that $\eta_0 = 0$, i.e., ζ_0 is real.

3. Let ζ_0 be real. Then in equation (30)

$$(\zeta_0+1)\int\limits_{(D_i)}\sum\left(\frac{\partial W^0}{\partial x}\right)^2 d\tau = (\zeta_0-1)\int\limits_{(D_e)}\sum\left(\frac{\partial W^0}{\partial x}\right)^2 d\tau$$

both integrals cannot vanish, for in this case $W^{(0)}$ and hence $\vartheta_0(0)$ would be identically equal to zero everywhere. If the integral on the right-hand side vanishes, then $\zeta_0 = -1$, and if the left-hand side vanishes, then it follows that $\zeta_0 = +1$. If both integrals are different from zero, then the quotient $\dfrac{\zeta_0 - 1}{\zeta_0 + 1}$ has a finite positive value, which means that the inequality $\zeta_0 > +1$ or $\zeta_0 < -1$ holds.

From this one obtains finally the result that there are no poles of the function μ between $\zeta_0 = -1$ and $\zeta_0 = +1$.

4. We now suppose that $k > 1$. Substituting the expansion of W into (B) and comparing the coefficients of $\dfrac{1}{(\zeta - \zeta_0)^{k-1}}$ on both sides, it follows that

$$\frac{dW_i^{(1)}}{dn} - \frac{dW_e^{(1)}}{dn} = -\zeta_0\left(\frac{dW_i^{(1)}}{dn} + \frac{dW_e^{(1)}}{dn}\right) - \left(\frac{dW_i^{(0)}}{dn} + \frac{dW_e^{(0)}}{dn}\right);$$

whence

$$(1+\zeta_0)\frac{dW_i^{(1)}}{dn} = (1-\zeta_0)\frac{dW_e^{(1)}}{dn} - \left(\frac{dW_i^{(0)}}{dn} + \frac{dW_e^{(0)}}{dn}\right). \tag{34}$$

We multiply (29) by $W^{(1)}$ and (34) by $W^{(0)}$ and then subtract the second product from the first. We thus obtain the equation

$$(1+\zeta_0)\left(W^{(1)}\frac{dW_i^{(0)}}{dn} - W^{(0)}\frac{dW_i^{(1)}}{dn}\right)$$
$$= (1-\zeta_0)\left(W^{(1)}\frac{dW_e^{(0)}}{dn} - W^{(0)}\frac{dW_e^{(1)}}{dn}\right) + \left(W^{(0)}\frac{dW_i^{(0)}}{dn} + W^{(0)}\frac{dW_e^{(0)}}{dn}\right).$$

Integrating this equation over (S), it follows that

$$\int\limits_{(D_i)}\sum\left(\frac{\partial W^{(0)}}{\partial x}\right)^2 d\tau - \int\limits_{(D_e)}\sum\left(\frac{\partial W^{(0)}}{\partial x}\right)^2 d\tau = 0. \tag{35}$$

We now write equation (30) in the form

$$(1+\zeta_0)\int\limits_{(D_i)}\sum\left(\frac{\partial W^{(0)}}{\partial x}\right)^2 d\tau + (1-\zeta_0)\int\limits_{(D_e)}\sum\left(\frac{\partial W^{(0)}}{\partial x}\right)^2 d\tau = 0. \tag{30''}$$

Since

$$\begin{vmatrix} 1 & -1 \\ 1+\zeta_0 & 1-\zeta_0 \end{vmatrix} = 1-\zeta_0+1+\zeta_0 = 2$$

we find by solving the system of equations arising from (35) and (30″):

$$\int_{(D_i)} \sum \left(\frac{\partial W^{(0)}}{\partial x} \right)^2 d\tau = 0, \qquad \int_{(D_e)} \sum \left(\frac{\partial W^{(0)}}{\partial x} \right)^2 d\tau = 0.$$

These equations can however not hold, for then $W^{(0)} \equiv 0$ and hence $\vartheta_0(0) \equiv 0$. Therefore the assumption that $k > 1$ leads to a contradiction; hence k can only be equal to one, as was to be shown. The four theorems of §5 have now been proved.

§8. The Necessary Condition that $\zeta = 1$ Not Be a Pole

If we substitute the value for μ of (17) into equation (7)

$$\mu(0) = -\frac{\zeta}{2\pi} \int_{(S_1)} \mu(1) \frac{\cos(r_{10} N_0)}{r_{10}^2} d\sigma_1 + f(0)$$

we obtain:

$$D_1(\zeta, 0) = -\frac{\zeta}{2\pi} \int_{(S_1)} D_1(\zeta, 1) \frac{\cos(r_{10} N_0)}{r_{10}^2} d\sigma_1 + f(0) D_1(\zeta). \tag{36}$$

We multiply this equation by $d\sigma$ and integrate over (S). We thus find:

$$\begin{aligned} \int_{(S)} D_1(\zeta, 0) d\sigma &= -\frac{\zeta}{2\pi} \int_{(S)} d\sigma \int_{(S_1)} D_1(\zeta, 1) \frac{\cos(r_{10} N_0)}{r_{10}^2} d\sigma_1 + D_1(\zeta) \int_{(S)} f(0) d\sigma \qquad (37) \\ &= -\frac{\zeta}{2\pi} \int_{(S_1)} D_1(\zeta, 1) \left(\int_{(S)} \frac{\cos(r_{10} N_0)}{r_{10}^2} d\sigma \right) d\sigma_1 + D_1(\zeta) \int_{(S)} f(0) d\sigma. \end{aligned}$$

Now since the point M_1 lies on (S),

$$\int_{(S)} \frac{\cos(r_{10} N_0)}{r_{10}^2} d\sigma = -\int_{(S)} \frac{\cos(r_{01} N_0)}{r_{01}^2} d\sigma = -2\pi.$$

Equation (37) now assumes the form

$$\int_{(S)} D_1(\zeta, 0) d\sigma = \zeta \int_{(S_1)} D_1(\zeta, 1) d\sigma_1 + D_1(\zeta) \int_{(S)} f(0) d\sigma$$

and it follows that

$$(1 - \zeta) \int_{(S)} D_1(\zeta, 0) d\sigma = D_1(\zeta) \int_{(S)} f(0) d\sigma. \tag{37'}$$

Let us suppose that the case at hand is the ordinary case or the case (J). For $\zeta = 1$ we obtain from (37′):

$$0 = D_1(1) \int\limits_{(S)} f(0)\, d\sigma. \tag{38}$$

From equation (38) it follows that either $\zeta = 1$ is a zero of the function $D_1(\zeta)$ or that

$$\int\limits_{(S)} f(0)\, d\sigma = 0. \tag{39}$$

Condition (39) is certainly satisfied if $\zeta = 1$ is not a zero of $D_1(\zeta)$. Hence, we have:

1. Equation (39) is the necessary condition that $\zeta = 1$ not be a pole of the function $\mu(0)$.

Assuming now that condition (39) is satisfied, we find from equation (37′):

$$(1-\zeta) \int\limits_{(S)} D_1(\zeta, 0)\, d\sigma = 0;$$

from this it follows

that for $\zeta \neq 1$

$$\int\limits_{(S)} D_1(\zeta, 0)\, d\sigma = 0.$$

The last integral is an entire function of ζ. If this function is zero for all values of ζ distinct from 1, then it is also zero for $\zeta = 1$. Hence,

2. If condition (39) is satisfied, then

$$\int\limits_{(S)} D_1(1, 0)\, d\sigma = 0. \tag{40}$$

We now suppose that $\zeta = 1$ is a zero of $D_1(\zeta)$. Dividing (37′) by $1 - \zeta$ and then putting $\zeta = 1$, we obtain:

$$\int\limits_{(S)} D_1(1, 0)\, d\sigma = -D_1'(1) \int\limits_{(S)} f(0)\, d\sigma. \tag{41}$$

But now μ has a simple pole at $\zeta = 1$. Therefore $D_1'(1)$ is different from zero, and we find:

3. If $\zeta = 1$ is a pole of $\mu(0)$, then the expression $\int\limits_{(S)} D_1(1,0) d\sigma$ vanishes only if condition (39) is satisfied.

In the case (E) condition (39) must be altered. Instead of integrating equation (37) over the entire boundary (S), we now integrate over each of the surfaces $(S^{(l)})$ $(l = 1, 2, \ldots, k)$ separately. We then obtain in place of (37) the equation

$$\begin{aligned}\int\limits_{(S^{(l)})} D_1(\zeta, 0)\, d\sigma = -\frac{\zeta}{2\pi} \int\limits_{(S_1)} D_1(\zeta, 1) \left(\int\limits_{(S^{(l)})} \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma \right) d\sigma_1 \\ + D_1(\zeta) \int\limits_{(S^{(l)})} f(0)\, d\sigma.\end{aligned} \tag{37$_1$}$$

Now

$$\int\limits_{(S^{(l)})} \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma = -\int\limits_{(S^{(l)})} \frac{\cos(r_{01} N_0)}{r_{10}^2}\, d\sigma;$$

this integral is different from zero only if the point M_1 belongs to the surface $(S^{(l)})$, since all point of the surfaces $(S^{(i)})$, $i \neq l$, lie outside the region bounded by $(S^{(l)})$. We may therefore write equations (37_1) in the form

$$\int\limits_{(S^{(l)})} D_1(\zeta, 0)\, d\sigma = \zeta \int\limits_{(S^{(l)})} D_1(\zeta, 0)\, d\sigma + D_1(\zeta) \int\limits_{(S^{(l)})} f(0)\, d\sigma$$

$$(l = 1, 2, \ldots, k)$$

From this follows

$$(1-\zeta) \int\limits_{(S^{(l)})} D_1(\zeta, 0)\, d\sigma = D_1(\zeta) \int\limits_{(S^{(l)})} f(0)\, d\sigma \qquad (37_1')$$

$$(l = 1, 2, \ldots, k).$$

Repeating now the considerations applied to the ordinary case and the case (J), we find:

1. The k equations

$$\int\limits_{(S^{(l)})} f(0)\, d\sigma = 0 \qquad (l = 1, 2, \ldots, k) \qquad (39')$$

are the necessary condition that $\zeta = 1$ not be a pole of the function $\mu(0)$.

2. If the k conditions (39′) are satisfied, then the k equations

$$\int\limits_{(S^{(l)})} D_1(1, 0)\, d\sigma = 0 \qquad (l = 1, 2, \ldots, k) \qquad (40')$$

also hold.

If $\zeta = 1$ is a pole of the function $\mu(0)$, then

$$\int\limits_{(S^{(l)})} D_1(1, 0)\, d\sigma = -D_1'(1) \int\limits_{(S^{(l)})} f(0)\, d\sigma \qquad (l = 1, 2, \ldots, k). \qquad (41')$$

From these equations we conclude:

3. If $\zeta = 1$ is a pole of $\mu(0)$, then the integral

$$\int\limits_{(S^{(l)})} D_1(1, 0)\, d\sigma$$

for a particular l is equal to zero only if for this value of l

$$\int\limits_{(S^{(l)})} f(0)\, d\sigma = 0.$$

§9. The Sufficiency of the Conditions Found

Lemma. *If the function* $\varrho(0)$ *satisfies the equation*

$$\varrho(0) = -\frac{1}{2\pi}\int_{(S_1)} \varrho(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1, \tag{42}$$

i.e., if $\varrho(0)$ *is an eigenfunction of equation (7) corresponding to the pole* $\zeta = 1$, *then the potential*

$$V = \int_{(S_1)} \frac{\varrho(1)\,d\sigma_1}{r_{10}} \tag{43}$$

is constant in the region (D_i).

Proof. Making use of (42), we find:

$$\frac{dV_i}{dn} - \frac{dV_e}{dn} = 4\pi\varrho(0) = -2\int_{(S_1)} \varrho(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1$$

$$= -2\frac{dV}{dn} = -\left(\frac{dV_i}{dn} + \frac{dV_e}{dn}\right).$$

From this it follows that

$$\frac{dV_i}{dn} = 0, \qquad \int_{(D_i)} \sum\left(\frac{\partial V}{\partial x}\right)^2 d\tau = 0,$$

i.e., in (D_i) $V = C$, as was to be proved.

In case (E) the constant has a particular value inside each of the regions bounded by the surfaces $(S^{(l)})$ which may be different from one region to the next. Hence, in case (E) one must write the equation obtained in the form $V = C^{(l)}$ if the point M_1 lies inside the region bounded by $(S^{(l)})$ $(l = 1, 2, \ldots, k)$.

To establish the sufficiency of conditions (39) and (39′) we consider the ordinary case with case (J) and the case (E) separately.

1. We first consider the ordinary case or the case (J) and suppose that condition (39) is satisfied:

$$\int_{(S)} f(0)\,d\sigma = 0.$$

Let $\zeta = 1$ be a pole of the function μ. In this case equation (40) holds. From equation (36)

$$D_1(\zeta, 0) = -\frac{\zeta}{2\pi}\int_{(S_1)} D_1(\zeta, 1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1 + D_1(\zeta)f(0),$$

we obtain, putting $\zeta = 1$, the equation

$$D_1(1, 0) = -\frac{1}{2\pi}\int_{(S_1)} D_1(1, 1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1, \tag{44}$$

i.e., $D_1(1,0)$ is a solution of equation (42). Hence, from the lemma

$$V = \int\limits_{(S_1)} \frac{D_1(1,1)\, d\sigma_1}{r_{10}} = C \quad \text{in} \quad (D_i). \tag{45}$$

We introduce a new function f_0 of the points of (S) and put

$$f_0(0) \equiv 1.$$

Then

$$\int\limits_{(S)} f_0(0)\, d\sigma = S, \tag{46}$$

where S denotes the sum of the areas of all boundary surfaces belonging to (S). If $\overline{\mu(0)}$ is the value of $\mu(0)$ corresponding to $f_0(0)$, then

$$\overline{\mu(0)} = \frac{\overline{D_1(\zeta, 0)}}{\overline{D_1(\zeta)}},$$

where $\overline{D_1(\zeta,0)}$ and $\overline{D_1(\zeta)}$ are certain entire functions obtained by replacing $f(0)$ in (17) by $f_0(0)$.

From (46) it follows that $\zeta = 1$ is a pole of the function $\overline{\mu(0)}$; we have:

$$\int\limits_{(S)} \overline{D_1(1, 0)}\, d\sigma = -\overline{D_1'(1)}\, S.$$

The equation

$$\overline{D_1(\zeta, 0)} = -\frac{\zeta}{2\pi} \int\limits_{(S_1)} \overline{D_1(\zeta, 1)} \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1 + \overline{D_1(\zeta)} f_0(0)$$

leads to the relation

$$\overline{D_1(1, 0)} = -\frac{1}{2\pi} \int\limits_{(S_1)} \overline{D_1(1, 1)} \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1,$$

from which it follows that $\overline{D_1(1,0)}$ is also a solution of equation (42). From this it then follows that

$$V_0 = \int\limits_{(S_1)} \frac{\overline{D_1(1, 1)}}{r_{10}}\, d\sigma_1 = C_0 \quad \text{in} \quad (D_i). \tag{45_1}$$

We multiply $\overline{D_1(1,0)}$ by $V d\sigma$ and integrate the product over (S). Since V is equal to C on (S), we find:

$$\int\limits_{(S)} \overline{D_1(1, 0)}\, V\, d\sigma = C \int\limits_{(S)} \overline{D_1(1, 0)}\, d\sigma = -\overline{D_1'(1)}\, S C.$$

On the other hand, from equations (40) and (45_1) we obtain:

$$\int_{(S)} \overline{D_1(1,0)}\, V\, d\sigma = \int_{(S)} \overline{D_1(1,0)} \left(\int_{(S_1)} \frac{D_1(1,1)\, d\sigma_1}{r_{10}} \right) d\sigma$$

$$= \int_{(S_1)} D_1(1,1) \left(\int_{(S)} \frac{\overline{D_1(1,0)}\, d\sigma}{r_{10}} \right) d\sigma_1 = C_0 \int_{(S_1)} D_1(1,1)\, d\sigma_1 = 0.$$

Hence,

$$-\overline{D_1'(1)}\, S\, C = 0, \qquad C = 0.$$

But if the potential of a simple layer is equal to zero in (D_i), then it vanishes everywhere; hence,

$$D_1(1,0) \equiv 0,$$

from which it follows that $\zeta = 1$ is a zero of the function $D_1(\zeta,0)$. But $\zeta = 1$ is also a zero of $D_1(\zeta)$; the fraction $\dfrac{D_1(\zeta,0)}{D_1(\zeta)}$ is then reducible, contrary to hypothesis. This then shows that the two hypotheses

1. $\zeta = 1$ is a zero of $D_1(\zeta)$,
2. condition (39) is satisfied,

contradict one another, i.e., $\zeta = 1$ cannot be a pole of $\mu(0)$.

In order to prove that in case (E) conditions (39′) are sufficient, we have to alter the considerations above somewhat. In this case we assume that the k conditions (39′) are satisfied and that $\zeta = 1$ is a pole of $\mu(0)$. Equations (40′) then hold, and we conclude from (36) that

$$V = \int_{(S_1)} \frac{D_1(1,1)\, d\sigma_1}{r_{10}} = C^{(l)} \quad \text{inside} \quad (S^{(l)}) \ \ (l = 1, 2, \ldots, k). \tag{45′}$$

We introduce a new function of the points of (S), putting

$$f_0(0) = \alpha^{(l)} \ \text{on} \ \ (S^{(l)}) \quad (l = 1, 2, \ldots, k),$$

where the $\alpha^{(l)}$ are arbitrarily chosen positive numbers.

We have:

$$\int_{(S)} f_0(0)\, d\sigma = \sum_{l=1}^{k} \int_{(S^{(l)})} f_0(0)\, d\sigma = \sum_{l=1}^{k} \alpha^{(l)} S^{(l)} > 0. \tag{46′}$$

If $\overline{\mu(0)}$ is the function μ corresponding to $f_0(0)$, then we may write

$$\overline{\mu(0)} = \frac{\overline{D_1(\zeta, 0)}}{\overline{D_1(\zeta)}},$$

where $\overline{D_1(\zeta,0)}$ and $\overline{D_1(\zeta)}$ are entire functions of ζ. On the basis of (46′), $\zeta = 1$ is certainly a zero of $\overline{D_1(\zeta)}$. Moreover,

$$\int_{(S^{(l)})} \overline{D_1(1,0)}\, d\sigma = -\overline{D_1'(1)}\, \alpha^{(l)} S^{(l)} \qquad (l = 1, 2, \ldots, k)$$

and

$$\overline{D_1(1,0)} = -\frac{1}{2\pi}\int_{(S_1)} \overline{D_1(1,1)}\,\frac{\cos(r_{10}\,N_0)}{r_{10}^2}\,d\sigma_1 .$$

Hence,

$$V_0 = \int_{(S_1)} \frac{\overline{D_1(1,1)}}{r_{10}}\,d\sigma_1 = C_0^{(l)} \quad \text{inside} \quad (S^{(l)}) \qquad (46'')$$

$$(l = 1, 2, \ldots, k).$$

From which we conclude that

$$\int_{(S)} \overline{D_1(1,0)}\,V\,d\sigma = \sum_{l=1}^{k} \int_{(S^{(l)})} \overline{D_1(1,0)}\,V\,d\sigma$$

$$= \sum_{l=1}^{k} C^{(l)} \int_{(S^{(l)})} \overline{D_1(1,0)}\,d\sigma = -\overline{D_1'(1)} \sum_{l=1}^{k} C^{(l)} \alpha^{(l)} S^{(l)} .$$

Then

$$\int_{(S)} \overline{D_1(1,0)}\,V\,d\sigma = \int_{(S)} \overline{D_1(1,0)} \left(\int_{(S_1)} \frac{D_1(1,1)\,d\sigma_1}{r_{10}} \right) d\sigma$$

$$= \int_{(S_1)} D_1(1,1) \left(\int_{(S)} \frac{\overline{D_1(1,0)}\,d\sigma}{r_{10}} \right) d\sigma_1$$

$$= \sum_{l=1}^{k} \int_{(S_1^{(l)})} D_1(1,1) \left(\int_{(S)} \frac{\overline{D_1(1,0)}\,d\sigma}{r_{10}} \right) d\sigma_1$$

$$= \sum_{l=1}^{k} C_0^{(l)} \int_{(S_1^{(l)})} D_1(1,1)\,d\sigma_1 = 0 .$$

Hence, for arbitrary positive

$$\sum_{l=1}^{k} C^{(l)} \alpha^{(l)} S^{(l)} = 0 ,$$

which is possible only if

$$C^{(l)} = 0 \qquad (l = 1, 2, \ldots, k)$$

From the last equations it follows that V is zero in (D_i) and hence vanishe. everywhere; hence,

$$D_1(1,0) \equiv 0 .$$

This equation, however, cannot hold, since it implies that the nominator and denumerator of the function $\mu(0)$ have a common factor. Therefore, condition (39′) and the hypothesis $D_1(1) = 0$ are also contradictory in case (E).

§10. The Solution of the Inner NEUMANN Problem

Ordinary case. We note first of all that in investigating the inner NEUMANN problem the function f cannot be prescribed arbitrarily.

Since a constant is a harmonic function in (D_i), it follows from formula (18) of §6 that

$$\int\limits_{(S)} \frac{d V_i}{d n} d\sigma = 0 .$$

From this it follows that in prescribing the function by the condition

$$\frac{d V_i}{d n} = f$$

equation (39) must also hold. Equation (39) is therefore a necessary condition that the problem have a solution.

In the previous section we have seen that if condition (39) is satisfied $\zeta = 1$ is not a pole of the meromorphic function (9):

$$\mu(0) = \frac{D_1(\zeta, 0)}{D_1(\zeta)} = \varrho_0 + \zeta \varrho_1 + \zeta^2 \varrho_2 + \cdots$$

with

$$\varrho_0 = f, \ldots, \varrho_n = -\frac{1}{2\pi} \int\limits_{(S_1)} \varrho_{n-1}(1) \frac{\cos(r_{10} N_0)}{r_{10}^2} d\sigma_1 .$$

Now $\zeta = 1$ is also not a pole of the function (11):

$$W = -\frac{1}{2\pi} \int\limits_{(S_1)} \frac{\mu(1)\, d\sigma_1}{r_{10}} = V_1 + \zeta V_2 + \zeta^2 V_3 + \cdots$$

with

$$V_k = -\frac{1}{2\pi} \int\limits_{(S_1)} \frac{\varrho_{k-1}(1)\, d\sigma_1}{r_{10}} = -\frac{1}{2\pi} \int\limits_{(S_1)} \frac{d V_{k-1}}{d n} \frac{d\sigma_1}{r_{10}} .$$

Functions (9) and (11) have no poles between $\zeta = -1$ and $\zeta = +1$. Since $\zeta = -1$ is also not a pole of these functions, we may set $\zeta = 1$. We find:

$$W = V_1 + V_2 + V_3 + \cdots, \tag{47}$$

where according to the remarks of §2

$$\frac{d W_i}{d n} = -f .$$

A solution of the problem is therefore given by the series

$$V = -V_1 - V_2 - V_3 - \cdots . \tag{47'}$$

Since the series

$$\mu_1(0) = \varrho_0 + \varrho_1 + \varrho_2 + \cdots \tag{9'}$$

converges uniformly on (S),
$$V = \frac{1}{2\pi}\int\limits_{(S)} \frac{\mu_1(0)\,d\sigma}{r_{10}}\,.$$

The potential V does indeed possess a normal derivative which satisfies the condition
$$\frac{dV_i}{dn} = f\,,$$

since the function (9′) is the solution of equation (7) for $\zeta = 1$.

Since a constant is a harmonic function in (D_i) whose normal derivative vanishes on (S), in solving the inner problem we may add an arbitrary constant to the function V. This remark is completely in accordance with the theorems proved above. One may clearly add an arbitrary solution of the homogeneous equations (42) to each solution μ of equation (7); but from the lemma in §9 this amounts to adding a constant to the function V.

Case (J). Just as in the ordinary case, in case (J) equation (39) is a necessary condition that the inner problem have a solution. The above considerations for the ordinary case are also valid for the case (J).

However, in case (J) the function (9) can have two poles: $\zeta = 1$ and $\zeta = -1$; if condition (39) is satisfied only the pole at $\zeta = 1$ vanishes. We cannot now find the value of the function W by simply putting $\zeta = 1$ in the series (11), since the radius of convergence of the series may be equal to one.

If we multiply the function W by $1 + \zeta$ we obtain:

$$(1+\zeta)\,W = V_1 + (V_2 + V_1)\zeta + (V_3 + V_2)\zeta^2 + \cdots + (V_n + V_{n-1})\zeta^{n-1} + \cdots.$$

If W has a pole at $\zeta = -1$, then since this pole is simple the function $(1 + \zeta)W$ has no pole there; $\zeta = 1$ is likewise not a pole of this function. Hence, the radius of convergence of the last series is always greater than one, so that we may set $\zeta = 1$. We find:

$$W = \tfrac{1}{2}\,[V_1 + (V_2 + V_1) + (V_3 + V_2) + \cdots + (V_n + V_{n-1}) + \cdots]. \tag{48}$$

The solution sought is given by the series

$$V = -\tfrac{1}{2}\,[V_1 + (V_2 + V_1) + (V_3 + V_2) + \cdots + (V_n + V_{n-1}) + \cdots]\,. \tag{48′}$$

As in the ordinary case, it is easily seen that V is indeed the potential of a simple layer and has a normal derivative which fulfills the conditions of the problem.

The remark above on the most general solution of the problem is also applicable here; to obtain this solution one adds an arbitrary constant to the function found.

Case (E). In case (E) one can apply formula (18) to each region bounded by

a surface $(S^{(l)})$. From this it follows that the inner NEUMANN problem has a solution only if the k conditions (39′)

$$\int\limits_{(S^{(l)})} f\,d\sigma = 0 \qquad (l = 1, 2, \ldots, k)$$

are satisfied.

If the k conditions (39′) are fulfilled, then $\zeta = 1$ is not a pole of the functions (9) and (11). In case (E) $\zeta = -1$ is likewise not a pole of these functions which also have no poles between $\zeta = -1$ and $\zeta = +1$. The radius of convergence of the series (11) is therefore greater than one; if we put $\zeta = 1$ there, then

$$W = V_1 + V_2 + V_3 + \cdots. \tag{49}$$

A solution of the NEUMANN problem is given by the series

$$V = -V_1 - V_2 - V_3 - \cdots \quad . \tag{49′}$$

We indicated that the treatment of the case at hand can be replaced by the investigation of k inner NEUMANN problems. The function which we would thus obtain as solution for the region bounded by $(S^{(l)})$ would be valid only inside this region, while the solution just found presents the function sought as a single series which is valid for all the inner regions.

One may add to the solution V found a function which is constant in each of the regions bounded by the surfaces $(S^{(l)})$ but whose value may change when passing from one region to another. This situation is in accordance with the lemma of §9 for the case (E); adding solutions of the homogeneous equation to the function μ amounts to simply adding constants to the potential V which have the property mentioned above.

We now wish to include a remark on the relation between problems A and B. We obtained condition (39) as a necessary and sufficient condition for the solvability of problem B. It is also therefore a sufficient condition that problem A have a solution. Is it sufficient? We shall show that this is the case. In order to have a specific case in mind, we consider the ordinary case.

For the proof we assume that the function U, harmonic in the interior of (D_i), satisfies condition (1) on the boundary surface (S), where

$$\int\limits_{(S)} f\,d\sigma = cS, \qquad c \neq 0.$$

The function $f_1 = f - c$ then satisfies condition (39), such that there exists a potential V of the simple layer having $\dfrac{dV_i}{dn} = f_1$. The functions $W = U - V$, harmonic in the interior of (D_i), then satisfies the condition

$$\frac{dW_i}{dn} = c$$

on (S). If $c > 0$ this means that the function W decreases as one moves from a boundary point along the normal into the interior of region (D_i). Hence, the point at which the function W assumes its minimum cannot lie on the boundary of the region (D_i); this contradicts a fundamental property of harmonic functions.

In the same way one sees that c cannot be less than zero. It is herewith shown that condition (39) is necessary for the existence of a solution of problem A.

From what has just been said it follows that every solution of problem A can be represented as the potential of a simple layer; this demonstrates the equivalence of problems A and B.

§11. The Solution of the Outer NEUMANN Problem for the Case (E) and for the Ordinary Case

We obtain the solution of the outer NEUMANN problem by determining the value of the function W for $\zeta = -1$. In case (E) and in the ordinary case the function W does not have a pole here. On the other hand, $\zeta = 1$ may be a pole, since the values of f in the outer problem are not limited by a particular condition. Hence, one may not set $\zeta = -1$ in the series (11).

If $\zeta = 1$ is a pole of W, then this pole is simple. The function $(1 - \zeta)W$ is therefore holomorphic in a neighborhood of $\zeta = 1$, and one may find the desired value of W by putting $\zeta = -1$ in the series

$$(1-\zeta)W = V_1 + (V_2 - V_1)\zeta + (V_3 - V_2)\zeta^2 + \cdots + (V_n - V_{n-1})\zeta^{n-1} + \cdots .$$

We then find that the function V satisfying the condition

$$\frac{dV_e}{dn} = f \quad \text{on} \quad (S)$$

can be represented by the series

$$V = \tfrac{1}{2}[V_1 - (V_2 - V_1) + (V_3 - V_2) - \cdots + (-1)^{n-1}(V_n - V_{n-1}) + \cdots] . \tag{50}$$

We shall treat the solution of the outer NEUMANN problem for the case (J) in §16. In this case $\zeta = -1$ may be a pole of the function W, so that the series (50) is no longer meaningful.

§12. The Eigenfunctions Corresponding to the Pole $\zeta = 1$ in the Ordinary Case and the ROBIN Problem for this Case

In the ordinary case $\zeta = -1$ is not a pole of the function $\mu(0)$. If condition (39)

$$\int_{(S)} f\,d\sigma = 0$$

is satisfied, then neither is $\zeta = 1$ a pole of this function. The radius of convergence of series (9)

$$\mu(0) = \varrho_0 + \zeta\varrho_1 + \cdots + \zeta^n\varrho_n + \cdots$$

is therefore greater than one. It is therefore possible to find a number τ in the interval $0 < \tau < 1$ such that series (9) converges for $\zeta = \frac{1}{\tau}$, and the general term

$$\frac{\varrho_n}{\tau^n}$$

of the transformed series is arbitrarily small in absolute value.

If condition (39) holds, then

$$|\varrho_n| < \tau^n \quad \text{for } n \geq N. \tag{51}$$

If the function f is prescribed arbitrarily, then the radius of convergence of series (9) may be equal to one. But since $\zeta = -1$ is not a pole of the function μ and since the pole at $\zeta = 1$ is simple, the series

$$(1-\zeta)\mu(0) = \varrho_0 + (\varrho_1 - \varrho_0)\zeta + \cdots + (\varrho_n - \varrho_{n-1})\zeta^n + \cdots$$

always has a radius of convergence greater than one, and there exists a positive number $\tau < 1$ such that

$$\frac{\varrho_n - \varrho_{n-1}}{\tau^n}$$

becomes arbitrarily small in absolute value.

Hence, for any arbitrary function f

$$|\varrho_n - \varrho_{n-1}| < \tau^n \quad \text{for} \quad n \geq N \qquad (0 < \tau < 1). \tag{52}$$

Since for $m > n$ the equation

$$\varrho_m - \varrho_n = (\varrho_m - \varrho_{m-1}) + (\varrho_{m-1} - \varrho_{m-2}) + \cdots + (\varrho_{n+1} - \varrho_n)$$

holds, we obtain the inequality

$$|\varrho_m - \varrho_n| \le |\varrho_m - \varrho_{m-1}| + \cdots + |\varrho_{n+1} - \varrho_n|$$
$$< \tau^{n+1} + \cdots + \tau^m < \frac{\tau^{n+1}}{1-\tau} < a\tau^n;$$

therefore, for any function f

$$|\varrho_m - \varrho_n| < a\tau^n \quad \text{for } m > n \ge N \quad (0 < \tau < 1). \tag{53}$$

The last inequality is called the *principle of* ROBIN. It shows that as $n \to \infty$ ϱ_n tends to a limit which we shall denote by ϱ:

$$\lim_{n\to\infty} \varrho_n = \varrho.$$

From inequality (51) it follows that if condition (39) is satisfied ϱ is equal to zero.

It is to be noted that the functions ϱ_n converge uniformly on (S), for in deriving inequality (53) no assumptions were made on the location of the point on (S).

Because of the uniform convergence of ϱ_n in (10) we may take the limit under the integral sign and write:

$$\varrho(0) = -\frac{1}{2\pi}\int_{(S_1)} \varrho(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1; \tag{54}$$

hence, if the limit function ϱ does not vanish it is an eigenfunction of equation (7),

$$\mu(0) = -\frac{\zeta}{2\pi}\int_{(S_1)} \mu(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1 + f(0),$$

corresponding to the pole $\zeta = 1$.

Integrating the nth equation in (10) over (S), we obtain:

$$\int_{(S)} \varrho_n(0)\,d\sigma = -\frac{1}{2\pi}\int_{(S)}\left(\int_{(S_1)} \varrho_{n-1}(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1\right)d\sigma$$
$$= -\frac{1}{2\pi}\int_{(S_1)} \varrho_{n-1}(1)\left(\int_{(S)}\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma\right)d\sigma_1$$
$$= \frac{1}{2\pi}\int_{(S_1)} \varrho_{n-1}(1)\left(\int_{(S)}\frac{\cos(r_{01}N_0)}{r_{10}^2}\,d\sigma\right)d\sigma_1$$
$$= \int_{(S_1)} \varrho_{n-1}(1)\,d\sigma_1.$$

Hence,

$$\int_{(S)} \varrho_n(0)\,d\sigma = \int_{(S)} \varrho_{n-1}(0)\,d\sigma = \cdots = \int_{(S)} \varrho_0(0)\,d\sigma = \int_{(S)} f(0)\,d\sigma.$$

Passing to the limit $n \to \infty$ in this equation, we find:

$$\int_{(S)} \varrho(0)\, d\sigma = \int_{(S)} f(0)\, d\sigma .$$

From this it follows that if condition (39) is not satisfied then ϱ is different from zero; ϱ is therefore in this case an eigenfunction of equation (7).

The product $C\varrho$, where C is a constant, is likewise a solution of equation (54). We wish to show that this is the most general solution, i.e., there is no eigenfunction corresponding to the pole $\zeta = 1$ which does not depend linearly on ϱ. For this we suppose that two functions f' and f'' are related by the equation

$$\int_{(S)} f'\, d\sigma = \int_{(S)} f''\, d\sigma \ . \tag{55}$$

Let ϱ' be the solution corresponding to f':

$$\varrho' = \lim_{n\to\infty} \varrho_n' \qquad \varrho_0' = f' ,$$

$$\varrho_n' = -\frac{1}{2\pi}\int_{(S_1)} \varrho_{n-1}' \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1 ;$$

similarly, let ϱ'' be the solution corresponding to f''. We shall find the solution corresponding to the function $f' - f''$; we denote it by ϱ. We have:

$$\varrho = \lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} (\varrho_n' - \varrho_n'')$$

with $\varrho_0 = f' - f'' = \varrho_0' - \varrho_0''$.

The equation $\varrho = \varrho' - \varrho''$ now follows. However, $\varrho = 0$, for from (55)

$$\int_{(S)} (f' - f'')\, d\sigma = 0 ;$$

hence,

$$\varrho' = \varrho'' ,$$

i.e., the two distinct functions f' and f'' related by equation (55) lead to the same solution.

If the functions f' and f'' do not satisfy relation (55), then at least one of the two integrals in (55) is different from zero. Supposing this to be the case, let $\int_{(S)} f' d\sigma$ be different from zero; one can then find a number b which satisfies the equation

$$b \int_{(S)} f'\, d\sigma = \int_{(S)} f''\, d\sigma . \tag{55'}$$

The solution corresponding to bf' is equal to $b\varrho'$, for if f' is multiplied by b ϱ_n' is also multiplied by b. Hence,

$$b\varrho' = \varrho'' .$$

The method of STEKLOV-ROBIN with any choice of f therefore leads to solutions of equation (54) which differ at most by a constant factor. Since

$$\int_{(S)} \varrho\, d\sigma = \int_{(S)} f\, d\sigma,$$

this factor is completely determined by the value of

$$\int_{(S)} f\, d\sigma.$$

It remains to see if by the method under consideration one can obtain all the solutions of equation (54).

Let ϱ be a solution of (54). We put $f = \varrho$; then

$$\varrho_0 = \varrho,$$

$$\varrho_1 = -\frac{1}{2\pi}\int_{(S)} \varrho \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma = \varrho,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\varrho_n = -\frac{1}{2\pi}\int_{(S)} \varrho \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma = \varrho, \qquad \lim_{n\to\infty} \varrho_n = \varrho.$$

We therefore see that with the method of STEKLOV-ROBIN we do indeed obtain every solution of equation (54).

If ϱ satisfies equation (54), then the potential (43)

$$V = \int_{(S_1)} \frac{\varrho(1)\, d\sigma_1}{r_{10}}$$

is constant in the region (D_i) by the lemma in §9.

Taking this into account, we may assert that in determining ϱ we have also solved the following problem in electrostatics (*the Robin problem*)$_0$ *It is required to find the charge distribution on a surface* (S) *such that there is no force exerted at any point in the interior of the region enclosed by* (S).

From what has been said it also follows that the solution of this problem is completely determined if the total amount of electric charge on (S),

$$\int_{(S)} \varrho\, d\sigma = M,$$

is known. To find the solution in this case, it is sufficient to put

$$f = \frac{M}{S};$$

we then obtain:

$$\int_{(S)} \varrho\, d\sigma = \frac{M}{S}\int_{(S)} d\sigma = M.$$

§13. The Eigenfunctions Corresponding to the Pole $\zeta = 1$ in Case (J) and the ROBIN Problem for this Case

In case (J) the function μ can in general have poles at $\zeta = 1$ and at $\zeta = -1$.

We shall assume first of all that condition (39) is satisfied. In this case $\zeta = 1$ is not a pole of μ; if $\zeta = -1$ is a pole this pole must be simple. Hence, the function

$$(1 + \zeta)\mu = \varrho_0 + (\varrho_1 + \varrho_0)\zeta + (\varrho_2 + \varrho_1)\zeta^2 + \cdots \tag{56}$$

has no pole at either $\zeta = 1$ or $\zeta = -1$, so that the radius of convergence of the series (56) is always greater than one, and the relation

$$\frac{\varrho_n + \varrho_{n-1}}{\tau^n}$$

for a certain positive value of τ becomes arbitrarily small in absolute value as $n \to \infty$; we have:

$$|\varrho_n + \varrho_{n-1}| < \tau^n \quad \text{for} \quad n \geq N.$$

From this inequality it follows that if condition (39) holds, then $\lim (\varrho_n + \varrho_{n-1}) = 0$.

Now let f be an arbitrary function. If μ has a pole at $\zeta = 1$, then this pole is simple; neither $\zeta = 1$ nor $\zeta = -1$ is then a pole of the function

$$\begin{aligned}(1 + \zeta)(1 - \zeta)\mu = \varrho_0 + \varrho_1\zeta & \\ + \{(\varrho_2 + \varrho_1) - (\varrho_1 + \varrho_0)\}\zeta^2 + \cdots & \\ + \{(\varrho_n + \varrho_{n-1}) - (\varrho_{n-1} + \varrho_{n-2})\}\zeta^n + \cdots .&\end{aligned} \tag{57}$$

Since the radius of convergence of the series (57) is always greater than one, for a certain number τ of the interval $0 < \tau < 1$ we have:

$$|(\varrho_n + \varrho_{n-1}) - (\varrho_{n-1} + \varrho_{n-2})| < \tau^n \quad \text{for} \quad n \geq N.$$

Using the same arguments as in §12, we obtain in analogy to inequality (53) the inequality

$$|(\varrho_m + \varrho_{m-1}) - (\varrho_n + \varrho_{n-1})| < a\tau^n \quad \text{for} \quad m > n \geq N. \tag{58}$$

From this inequality it follows that

$$\varrho_n + \varrho_{n-1} \tag{59}$$

tends to a limit which we shall denote by ϱ. It is clear that the quantity (59) converges uniformly on (S). We have now that

$$\left.\begin{aligned}\varrho_n(0) &= -\frac{1}{2\pi}\int\limits_{(S_1)}\varrho_{n-1}(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1,\\ \varrho_{n-1}(0) &= -\frac{1}{2\pi}\int\limits_{(S_1)}\varrho_{n-2}(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1\end{aligned}\right\}, \tag{60}$$

and hence

$$\varrho_n(0)+\varrho_{n-1}(0) = -\frac{1}{2\pi}\int\limits_{(S_1)}[\varrho_{n-1}(1)+\varrho_{n-2}(1)]\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1 .$$

Passing to the limit $n\to\infty$, we find:

$$\varrho(0) = -\frac{1}{2\pi}\int\limits_{(S_1)}\varrho(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1 . \tag{61}$$

If then the limit function is nonzero it is an eigenfunction of equation (7) corresponding to the pole $\zeta = 1$. Integrating the first equation in (60), we obtain:

$$\begin{aligned}\int\limits_{(S)}\varrho_n\,d\sigma &= -\frac{1}{2\pi}\int\limits_{(S)}\left(\int\limits_{(S_1)}\varrho_{n-1}(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1\right)d\sigma\\ &= \frac{1}{2\pi}\int\limits_{(S_1)}\varrho_{n-1}(1)\left(\int\limits_{(S)}\frac{\cos(r_{01}N_0)}{r_{10}^2}\,d\sigma\right)d\sigma_1 = \int\limits_{(S_1)}\varrho_{n-1}(1)\,d\sigma_1 .\end{aligned}$$

From this it follows that

$$\int\limits_{(S)}\varrho_n\,d\sigma = \int\limits_{(S)}\varrho_{n-1}\,d\sigma = \cdots = \int\limits_{(S)}\varrho_0\,d\sigma = \int\limits_{(S)} f\,d\sigma ,$$

$$\int\limits_{(S)}(\varrho_n+\varrho_{n-1})\,d\sigma = 2\int\limits_{(S)} f\,d\sigma .$$

Passing to the limit gives:

$$\int\limits_{(S)}\varrho\,d\sigma = 2\int\limits_{(S)} f\,d\sigma .$$

From this equation we can conclude that $\varrho \neq 0$ if condition (39) is not satisfied. Integrating (60) over one of the inner surfaces $(S^{(l)})$, we find:

$$\begin{aligned}\int\limits_{(S^{(l)})}\varrho_n\,d\sigma &= -\frac{1}{2\pi}\int\limits_{(S^{(l)})}\left(\int\limits_{(S_1)}\varrho_{n-1}(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1\right)d\sigma\\ &= \frac{1}{2\pi}\int\limits_{(S_1)}\varrho_{n-1}(1)\left(\int\limits_{(S^{(l)})}\frac{\cos(r_{01}N_0)}{r_{10}^2}\,d\sigma\right)d\sigma_1\\ &= -\int\limits_{(S_1^{(l)})}\varrho_{n-1}(1)\,d\sigma_1 .\end{aligned}$$

Indeed, the integral

$$\int\limits_{(S^{(l)})} \frac{\cos(r_{01} N_0)}{r_{10}^2}\, d\sigma$$

is different from zero if the point M_1 lies on $(S^{(l)})$; in this case it is equal to -2π, since the normals are directed into the interior of $(S^{(l)})$. Hence,

$$\int\limits_{(S^{(l)})} \varrho_n\, d\sigma = (-1)\int\limits_{(S^{(l)})} \varrho_{n-1}\, d\sigma = \cdots = (-1)^n \int\limits_{(S^{(l)})} f\, d\sigma .$$

so that if for one of the surfaces $(S^{(l)})$ $(l = 1, 2, \ldots, k)$

$$\int\limits_{(S^{(l)})} f\, d\sigma \neq 0 ,$$

then ϱ_n has no limit, whence it follows that the principle of ROBIN is not satisfied.

The equations

$$\int\limits_{(S^{(l)})} (\varrho_n + \varrho_{n-1})\, d\sigma = 0 , \qquad \int\limits_{(S^{(l)})} \varrho\, d\sigma = 0 \qquad (l = 1, 2, \ldots, k)$$

follow from the formulas obtained. The last of these equations can be obtained from equation (61); indeed, integrating over $(S^{(l)})$, one finds:

$$\begin{aligned} \int\limits_{(S^{(l)})} \varrho\, d\sigma &= -\frac{1}{2\pi} \int\limits_{(S^{(l)})} \Bigg(\int\limits_{(S_1)} \varrho(1) \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1 \Bigg) d\sigma \\ &= -\frac{1}{2\pi} \int\limits_{(S_1)} \varrho(1) \Bigg(\int\limits_{(S^{(l)})} \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma \Bigg) d\sigma_1 = -\int\limits_{(S_1^{(l)})} \varrho(1)\, d\sigma_1 . \end{aligned}$$

We shall subsequently prove that ϱ is equal to zero on $(S^{(l)})$.

One easily sees that the general solution of equation (61) is obtained by multiplying the function ϱ found by an arbitrary constant.

If the functions f' and f'' are related by condition (55), then we find with the help of the considerations of the previous section that the solutions ϱ' and ϱ'' corresponding to these functions coincide. For the function $f' - f''$ satisfies condition (39), so that the corresponding solution $\varrho' - \varrho''$ is equal to zero. From this it follows that the function ϱ is completely determined by the condition

$$\int\limits_{(S)} \varrho\, d\sigma = M .$$

To determine ϱ by the method described, it suffices to put $f = \dfrac{M}{2S}$.

From arguments already presented in §12 we see that the method presented actually gives all the solutions of (61). Indeed, if ϱ is a particular solution of this equation and if we put $f = \dfrac{1}{2}\varrho$ then

$$\varrho_n = \tfrac{1}{2}\varrho, \qquad \varrho_n + \varrho_{n-1} = \varrho;$$

we thus return to the function ϱ.

From the lemma of §9 the potential

$$V = \int\limits_{(S_1)} \frac{\varrho(1)\, d\sigma_1}{r_{10}} \tag{62}$$

is constant in (D_i), so that in the case (J) in question we have found the solution of the ROBIN problem.

Integrating over a surface $(S^{(l)})$ $(l = 1, 2, \ldots, k)$, we obtain:

$$\int\limits_{(D_e^{(l)})} \sum \left(\frac{\partial V}{\partial x}\right)^2 d\tau = -\int\limits_{(S^{(l)})} V \frac{dV_e}{dn}\, d\sigma = -C \int\limits_{(S^{(l)})} \frac{dV_e}{dn}\, d\sigma = 0;$$

$(D_e^{(l)})$ here denotes the region bounded by $(S^{(l)})$.

Thus, V is also constant in the interior of $(D_e^{(l)})$. From this it follows that V retains the same constant value inside $(S^{(0)})$ and that on the surfaces $(S^{(l)})$ $(l = 1, 2, \ldots, k)$

$$4\pi\varrho = \frac{dV_i}{dn} - \frac{dV_e}{dn} = 0,$$

i.e., ϱ is equal to zero on the inner surfaces.

In the language of electrostatics this says that the total charge is distributed on the outer surface $(S^{(0)})$.

Because of this situation the function ϱ satisfies the equation

$$\varrho(0) = -\frac{1}{2\pi} \int\limits_{(S_1^{(0)})} \varrho(1) \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1; \tag{61'}$$

therefore, ϱ is the same function which we found in the previous section as solution of the ROBIN problem for the surface $(S^{(0)})$.

§14. The Eigenfunction Corresponding to the Pole $\zeta = 1$ in Case (E) and the ROBIN Problem for this Case

In case (E) $\zeta = -1$ is not a pole of the function μ. If the conditions (39′)

$$\int\limits_{(S^{(l)})} f\, d\sigma = 0 \qquad (l = 1, 2, \ldots, k)$$

are satisfied, then $\zeta = 1$ is also not a pole; hence, the radius of convergence of series (9) is greater than one.

If $\tau < 1$ is a positive number and $n \geqq N$, then in this case

$$|\varrho_n| < \tau^n, \qquad \lim_{n\to\infty} \varrho_n = 0.$$

If f is prescribed arbitrarily, the radius of convergence of the series

$$(1-\zeta)\,\mu(0) = \varrho_0 + (\varrho_1 - \varrho_0)\,\zeta + (\varrho_2 - \varrho_1)\,\zeta^2 + \cdots$$

is greater than one; hence,

$$|\varrho_n - \varrho_{n-1}| < \tau^n \quad \text{for } n \geq N$$

The considerations of §12 establish the validity of the inequality

$$|\varrho_m - \varrho_n| < a\tau^n \quad \text{for } m > n \geq N,$$

i.e., in the case at hand the principle of ROBIN is satisfied.

Passing to the limit $n \to \infty$ in the equation

$$\varrho_n(0) = -\frac{1}{2\pi}\int\limits_{(S_1)} \varrho_{n-1}(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1\,, \tag{63}$$

one obtains equation (61)

$$\varrho(0) = -\frac{1}{2\pi}\int\limits_{(S_1)} \varrho(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1,$$

i.e., if the function ϱ is nonzero it represents an eigenfunction of equation (7) corresponding to the pole $\zeta \doteq 1$.

Integrating (63) over $(S^{(l)})$, we find:

$$\begin{aligned}\int\limits_{(S^{(l)})} \varrho_n\,d\sigma &= -\frac{1}{2\pi}\int\limits_{(S^{(l)})}\left(\int\limits_{(S_1)} \varrho_{n-1}(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1\right)d\sigma \\ &= \frac{1}{2\pi}\int\limits_{(S_1)} \varrho_{n-1}(1)\left(\int\limits_{(S^{(l)})}\frac{\cos(r_{01}N_0)}{r_{10}^2}\,d\sigma\right)d\sigma_1 = \int\limits_{(S_1{}^{(l)})} \varrho_{n-1}(1)\,d\sigma_1\,;\end{aligned}$$

the integral

$$\int\limits_{(S^{(l)})}\frac{\cos(r_{01}N_0)}{r_{10}^2}\,d\sigma$$

does not vanish only if the point M_1 lies on $(S^{(l)})$; in this case it is equal to 2π. Hence,

$$\left.\begin{aligned}&\int\limits_{(S^{(l)})} \varrho_n\,d\sigma = \int\limits_{(S^{(l)})} \varrho_{n-1}\,d\sigma = \cdots = \int\limits_{(S^{(l)})} f\,d\sigma, \\ &\int\limits_{(S^{(l)})} \varrho\,d\sigma = \int\limits_{(S^{(l)})} f\,d\sigma \qquad (l = 1, 2, \ldots, k).\end{aligned}\right| \tag{64}$$

We thus see that ϱ is different from zero if one of the conditions (39′) is not satisfied.

From equation (64) it follows that there exist several linearly independent eigenfunctions. Indeed, the eigenfunction corresponding to the function f_1 with

$$\int\limits_{(S^{(1)})} f_1\,d\sigma \neq 0\,, \qquad \int\limits_{(S^{(2)})} f_1\,d\sigma = 0$$

is linearly independent of the eigenfunction corresponding to the function f_2 with

$$\int\limits_{(S^{(1)})} f_2\,d\sigma = 0\,, \qquad \int\limits_{(S^{(2)})} f_2\,d\sigma \neq 0\,.$$

We introduce k functions $f^{(1)}, f^{(2)}, \ldots, f^{(k)}$ which have the property that $f^{(l)} = 0$ if M_0 does not lie on $(S^{(l)})$ and $f^{(l)} = \frac{1}{S^{(l)}}$ if M_0 does lie on $(S^{(l)})$. Let the corresponding eigenfunctions be

$$\varrho^{(1)},\ \varrho^{(2)},\ \ldots,\ \varrho^{(l)}, \ldots, \varrho^{(k)}. \tag{65}$$

The function $\varrho^{(l)}$ satisfies the conditions

$$\left.\begin{aligned} &\int\limits_{(S^{(\lambda)})} \varrho^{(l)} d\sigma = 0 \quad \text{for} \quad \lambda \neq l, \\ &\int\limits_{(S^{(l)})} \varrho^{(l)} d\sigma = \frac{1}{S^{(l)}} \int\limits_{(S^{(l)})} d\sigma = 1. \end{aligned}\right\} \tag{66}$$

The eigenfunctions (65) are linearly independent, for if there exist constants $C_1, C_2, \ldots, C_k$ such that

$$C_1 \varrho^{(1)} + C_2 \varrho^{(2)} + \cdots + C_l \varrho^{(l)} + \cdots + C_k \varrho^{(k)} \equiv 0,$$

then integrating the last equation over $(S^{(l)})$ we find:

$$C_l = 0.$$

Every eigenfunction ϱ obtained from a certain function f in the manner described is a linear combination of the functions (65). Indeed, if

$$\int\limits_{(S^{(l)})} f \, d\sigma = \alpha^{(l)} \qquad (l = 1, 2, \ldots, k), \tag{67}$$

and ϱ is the eigenfunction corresponding to f, then the function

$$f' = f - \alpha^{(1)} f^{(1)} - \alpha^{(2)} f^{(2)} - \cdots - \alpha^{(k)} f^{(k)}$$

corresponds to the eigenfunction

$$\varrho - \alpha^{(1)} \varrho^{(1)} - \alpha^{(2)} \varrho^{(2)} - \cdots - \alpha^{(k)} \varrho^{(k)},$$

which vanishes since f' satisfies the k conditions (39′).

Finally, every solution of equation (61) can be obtained in this manner. Let ϱ be such a solution. Setting $f = \varrho$ we find

$$\varrho_0 = \varrho,\ \varrho_1 = \varrho, \ldots,\ \varrho_n = \varrho, \ldots;\ \lim_{n \to \infty} \varrho_n = \varrho.$$

From the foregoing one can further conclude that the solution of equation (61) is completely determined as soon as the values $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(k)}$ of the integral

$$\int\limits_{(S^{(l)})} \varrho \, d\sigma \qquad (l = 1, 2, \ldots, k)$$

are known; the solution then reads

$$\varrho = \alpha^{(1)}\varrho^{(1)} + \cdots + \alpha^{(k)}\varrho^{(k)}.$$

From the lemma of §9 we come to the conclusion that the function

$$V = \int\limits_{(S_1)} \frac{\varrho(1)\, d\sigma_2}{r_{10}}$$

solves the Robin problem in case (E) and that the solution of this problem is completely determined by the distribution of electric charge on each of the surfaces $(S^{(l)})$ $(l = 1, 2, \ldots, k)$.

§15. The Pole $\zeta = -1$ in Case (J)

We shall now return to the investigation of the function (17)

$$\mu(0) = \frac{D_1(\zeta, 0)}{D_1(\zeta)}$$

and determine under what conditions $\zeta = -1$ is not a pole of this function. Of the function $D_1(\zeta, 0)$ we know that it satisfies the equation (36)

$$D_1(\zeta, 0) = -\frac{\zeta}{2\pi}\int\limits_{(S_1)} D_1(\zeta, 1)\frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1 + D_1(\zeta) f(0)\,.$$

We integrate (36) over each of the inner surfaces

$$(S^{(1)}), \ldots, (S^{(k)}).$$

This gives:

$$\begin{aligned}\int\limits_{(S^{(l)})} D_1(\zeta, 0)\, d\sigma &= -\frac{\zeta}{2\pi}\int\limits_{(S^{(l)})}\left(\int\limits_{(S_1)} D_1(\zeta, 1)\frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1\right) d\sigma + D_1(\zeta)\int\limits_{(S^{(l)})} f(0)\, d\sigma \\ &= \frac{\zeta}{2\pi}\int\limits_{(S_1)} D_1(\zeta, 1)\left(\int\limits_{(S^{(l)})}\frac{\cos(r_{01} N_0)}{r_{10}^2}\, d\sigma\right) d\sigma_1 + D_1(\zeta)\int\limits_{(S^{(l)})} f(0)\, d\sigma \\ &= -\zeta\int\limits_{(S_1^{(l)})} D_1(\zeta, 1)\, d\sigma_1 + D_1(\zeta)\int\limits_{(S^{(l)})} f(0)\, d\sigma;\end{aligned}$$

the integral

$$\int\limits_{(S^{(l)})}\frac{\cos(r_{01} N_0)}{r_{10}^2}\, d\sigma$$

is different from zero if the point M_1 lies on $(S^{(l)})$; in this case it is equal to -2π, since the normals of the surface $(S^{(l)})$ are directed into the interior of the region bounded by this surface. For the function $D_1(\zeta, 0)$ it follows from the last equation that

$$(1+\zeta)\int\limits_{(S^{(l)})} D_1(\zeta, 0)\, d\sigma = D_1(\zeta)\int\limits_{(S^{(l)})} f(0)\, d\sigma \qquad (l = 1, 2, \ldots, k)\,. \tag{68}$$

Putting $\zeta = -1$ in this equation we obtain

$$0 = D_1(-1)\int_{(S^{(l)})} f(0)\, d\sigma \qquad (l = 1, 2, \ldots, k).$$

Thus, the k equations

$$\int_{(S^{(l)})} f(0)\, d\sigma = 0 \qquad (l = 1, 2, \ldots, k). \tag{69}$$

are the necessary condition that $\zeta = -1$ is not a pole of the function (17).

If the k conditions (69) are satisfied, then equations (68) have the form

$$(1 + \zeta)\int_{(S^{(l)})} D_1(\zeta, 0)\, d\sigma = 0;$$

hence, for $\zeta \neq -1$

$$\int_{(S^{(l)})} D_1(\zeta, 0)\, d\sigma = 0 \qquad (l = 1, 2, \ldots, k). \tag{70}$$

Since each of these integrals is an entire function of ζ, equations (70) must also be satisfied for $\zeta = -1$; hence if the k conditions (69) hold we find:

$$\int_{(S^{(l)})} D_1(-1, 0)\, d\sigma = 0 \qquad (l = 1, 2, \ldots, k). \tag{71}$$

We shall now assume that $\zeta = -1$ is a pole of the function (17). Dividing both sides of equation (68) by $1 + \zeta$ and then putting $\zeta = -1$, we find:

$$\int_{(S^{(l)})} D_1(-1, 0)\, d\sigma = D_1'(-1)\int_{(S^{(l)})} f\, d\sigma \qquad (l = 1, 2, \ldots, k).$$

From this it follows that if $\zeta = -1$ is a pole of (17) the equation

$$\int_{(S^{(l)})} D_1(-1, 0)\, d\sigma = 0$$

holds only if the condition

$$\int_{(S^{(l)})} f\, d\sigma = 0$$

is satisfied.

With considerations analogous to those of §9 we can easily show that $\zeta = -1$ is not a pole of (17) if conditions (69) are satisfied. We first prove the following lemma.

Lemma. *If the function ϱ satisfies the equation*

$$\varrho(0) = \frac{1}{2\pi} \int\limits_{(S_1)} \varrho(1) \frac{\cos(r_{10} N_0)}{r_{10}^2} d\sigma_1,$$

then the potential

$$V = \int\limits_{(S_1)} \frac{\varrho(1)\, d\sigma_1}{r_{10}}$$

is constant in the individual connected components of the region (D_e).

Proof. On each of the boundary surfaces $(S^{(0)}), \ldots, (S^{(k)})$

$$\frac{dV_i}{dn} - \frac{dV_e}{dn} = 4\pi\varrho(0) = 2\int\limits_{(S_1)} \varrho(1) \frac{\cos(r_{10} N_0)}{r_{10}^2} d\sigma_1 = \frac{dV_i}{dn} + \frac{dV_e}{dn};$$

hence,

$$\frac{dV_e}{dn} = 0,$$

and therefore

$$\int\limits_{(D_e)} \sum \left(\frac{\partial V}{\partial x}\right)^2 d\tau = 0.$$

Hence, V is constant in each connected subregion of (D_e).

It will now be assumed that the function f satisfies the k conditions (69) and that $\zeta = -1$ is a pole of (17). From this there follows the relation

$$D_1(-1, 0) = \frac{1}{2\pi} \int\limits_{(S_1)} D_1(-1, 1) \frac{\cos(r_{10} N_0)}{r_{10}^2} d\sigma_1$$

and the validity of equations (71).

From the lemma we conclude that

$$V = \int\limits_{(S_1)} \frac{D_1(-1, 1)}{r_{10}} d\sigma_1$$

is constant in (D_e). Let $V = C^{(l)}$ be the value of V in the region bounded by $(S^{(l)})$ $(l = 1, 2, \ldots, k)$. Outside $(S^{(0)})$ the function V is equal to zero, since it vanishes at infinity.

We introduce the function f_0 which satisfies the conditions

$$f_0 = \alpha^{(l)} \quad \text{on} \quad (S^{(l)}), \qquad (l = 1, 2, \ldots, k)$$

where the $\alpha^{(l)}$ are some nonzero constants. We have:

$$\int\limits_{(S^{(l)})} f_0\, d\sigma = \alpha^{(l)} S^{(l)};$$

hence, f_0 does not satisfy conditions (69), and $\zeta = -1$ is certainly a pole of the corresponding function

$$\frac{\overline{D_1(\zeta, 0)}}{D_1(\zeta)}. \tag{72}$$

But if now $\zeta = -1$ is a zero of $D_1(\zeta)$, then the function $\overline{D_1(\zeta, 0)}$ satisfies the equation

$$\overline{D_1(-1,0)} = \frac{1}{2\pi}\int\limits_{(S_1)} \overline{D_1(-1,1)}\,\frac{\cos(r_{10} N_0)}{r_{10}^2}\,d\sigma_1;$$

from this it follows that the potential

$$V_0 = \int\limits_{(S_1)} \frac{\overline{D_1(-1,1)}}{r_{10}}\,d\sigma_1$$

is constant in each connected component of (D_e). Let $C_0^{(l)}$ be the value of V_0 in the component of (D_e) bounded by $(S^{(l)})$ $(l = 1, 2, \ldots, k)$; outside $(S^{(0)})$ V_0 is equal to zero.

We multiply $\overline{D_1(-1,0)}$ by V and integrate the product over (S); it follows since V vanishes on $(S^{(0)})$ that

$$\int\limits_{(S)} \overline{D_1(-1,0)}\,V d\sigma = \sum_{l=1}^{k} C^{(l)} \int\limits_{(S^{(l)})} \overline{D_1(-1,0)}\,d\sigma$$
$$= \sum_{l=1}^{k} C^{(l)} \alpha^{(l)}\, \overline{D_1'(-1)}\, S^{(l)}.$$

On the other hand, we find with the help of equation (71):

$$\int\limits_{(S)} \overline{D_1(-1,0)}\,V d\sigma = \int\limits_{(S)} \overline{D_1(-1,0)} \left(\int\limits_{(S_1)} \frac{D_1(-1,1)}{r_{10}}\,d\sigma_1 \right) d\sigma$$
$$= \int\limits_{(S_1)} D_1(-1,1) \left(\int\limits_{(S)} \frac{\overline{D_1(-1,0)}}{r_{10}}\,d\sigma \right) d\sigma_1$$
$$= \int\limits_{(S_1)} D_1(-1,1)\, V_0(1)\, d\sigma_1$$
$$= \sum_{l=1}^{k} C_0^{(l)} \int\limits_{(S_1^{(l)})} D_1(-1,1)\,d\sigma_1 = 0.$$

Comparing the last two results, we see that the constants $\alpha^{(l)}$ are related by the relation

$$\sum_{l=1}^{k} C^{(l)} \alpha^{(l)}\, \overline{D_1'(-1)}\, S^{(l)} = 0.$$

From this it follows that

$$C^{(l)} = 0. \qquad (l = 1, 2, \ldots, k)$$

Hence, V is equal to zero in the entire region (D_e), which means that the function $D_1(-1,0)$ is everywhere zero. We have now obtained a contradiction, since it has been shown that the numerator and denominator of the function (17) must have a common factor contrary to the original hypothesis.

Hence, if conditions (69) hold, then $\zeta = -1$ cannot be a pole of (17).

§16. The Outer NEUMANN Problem for the Case (J)

The outer problem has a solution only if the equations

$$\int_{(S^{(l)})} f\, d\sigma = 0 \qquad (l = 1, 2, \ldots, k) \tag{69'}$$

hold for all the inner boundary surfaces. Indeed, the potential sought is harmonic inside each region bounded by an inner surface; if

$$f = \frac{dV_e}{dn} \qquad \text{on} \qquad (S^{(l)}),$$

then the equality

$$\int_{(S^{(l)})} f\, d\sigma = \int_{(S^{(l)})} \frac{dV_e}{dn}\, d\sigma = 0$$

must hold. If conditions (69′) hold $\zeta = -1$ is not a pole of the function

$$V = V_1 + V_2\zeta + V_3\zeta^2 + \cdots, \tag{73}$$

since $\zeta = -1$ is not a pole of μ.

If the function (73) has a pole at $\zeta = 1$, then this pole is simple. Hence, the radius of convergence of the series

$$(1-\zeta)V = V_1 + (V_2 - V_1)\zeta + (V_3 - V_2)\zeta^2 + \cdots \tag{74}$$

is always greater than one. One may therefore compute the value of V by putting $\zeta = -1$ in (74). The function

$$V = \tfrac{1}{2}\{V_1 - (V_2 - V_1) + (V_3 - V_2) - + \cdots\}$$

is the solution of the outer NEUMANN problem.

To the solution found a function α can be added which vanishes outside the outer boundary $(S^{(0)})$ and assumes constant values inside the inner surfaces. Indeed, one may add solutions of the homogeneous equation to the function μ which is equivalent to adding a function α of the form indicated to the potential V.

§17. The Eigenfunctions Corresponding to the Pole $\zeta = -1$ in the Case (J)

If the conditions (69)

$$\int_{(S^{(l)})} f\, d\sigma = 0 \qquad\qquad (l = 1, 2, \ldots, k)$$

are satisfied, then $\zeta = -1$ is not a pole of the function (9)

$$\mu(0) = \varrho_0(0) + \zeta\varrho_1(0) + \zeta^2\varrho_2(0) + \cdots.$$

Neither $\zeta = 1$ nor $\zeta = -1$ are then poles of the function

$$\mu(0)(1-\zeta) = \varrho_0 + (\varrho_1 - \varrho_0)\zeta + (\varrho_2 - \varrho_1)\zeta^2 + \cdots + (\varrho_n - \varrho_{n-1})\zeta^n + \cdots,$$

so that the radius of convergence of this series is always greater than one. For some positive number $\tau < 1$,

$$|\varrho_n - \varrho_{n-1}| < \tau^n \quad \text{for } n \geq N;$$

whence follows the inequality

$$|\varrho_m - \varrho_n| < a\tau^n \quad \text{for } m > n \geq N.$$

This inequality shows that if conditions (69) hold ϱ has a limit.

Let f be an arbitrary function. The function $\mu(0)$ may then have poles at $\zeta = 1$ and $\zeta = -1$. Since these poles must however be simple, the radius of convergence of the series

$$(1-\zeta^2)\mu(0) = \varrho_0 + \varrho_1\zeta + (\varrho_2 - \varrho_0)\zeta^2 + (\varrho_3 - \varrho_1)\zeta^3 + \cdots + (\varrho_n - \varrho_{n-2})\zeta^n + \cdots \tag{75}$$

is always greater than one. Hence, it is possible to find a positive number $\tau < 1$ such that

$$|\varrho_n - \varrho_{n-2}| < \tau^n \quad \text{for } n \geq N. \tag{76}$$

From this inequality it follows that if m and n are either both even or both odd and $m > n$, then

$$\begin{aligned}|\varrho_m - \varrho_n| &\leq |\varrho_m - \varrho_{m-2}| + |\varrho_{m-2} - \varrho_{m-4}| + \cdots + |\varrho_{n+2} - \varrho_n| \\ &< \tau^m + \tau^{m-2} + \cdots + \tau^{n+2} \\ &< a\tau^n.\end{aligned}$$

The sequences

$$\varrho_0, \varrho_2, \varrho_4, \ldots, \varrho_{2n}, \ldots, \tag{77'}$$

$$\varrho_1, \varrho_3, \varrho_5, \ldots, \varrho_{2n+1}, \ldots \tag{77''}$$

have limits which we denote by A and B.

If conditions (69) are satisfied, then ϱ has a limit; in this case $A = B$. Now

$$\left.\begin{aligned}\varrho_{2n}(0) &= -\frac{1}{2\pi}\int\limits_{(S_1)} \varrho_{2n-1}(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1, \\ \varrho_{2n-1}(0) &= -\frac{1}{2\pi}\int\limits_{(S_1)} \varrho_{2n-2}(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1.\end{aligned}\right| \tag{60'}$$

Passing to the limit, we find:

$$\left.\begin{aligned} A(0) &= -\frac{1}{2\pi}\int_{(S_1)} B(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1, \\ B(0) &= -\frac{1}{2\pi}\int_{(S_1)} A(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1; \end{aligned}\right| \tag{78}$$

subtracting these two equations it follows that

$$A(0) - \dot{B}(0) = \frac{1}{2\pi}\int_{(S_1)}[A(1) - B(1)]\frac{\cos(r_{10}N_0)}{r_{10}^2}\,d\sigma_1. \tag{79}$$

This implies that if the function

$$\psi = A - B, \tag{80}$$

is nonzero it is an eigenfunction of equation (7) corresponding to the pole $\zeta = -1$.

Integrating (60′) over $(S^{(l)})$ $(l = 1, 2, \ldots, k)$, we obtain just as in §13:

$$\int_{(S^{(l)})}\varrho_n\,d\sigma = -\int_{(S^{(l)})}\varrho_{n-1}\,d\sigma = \cdots = (-1)^n\int_{(S^{(l)})} f\,d\sigma;$$

then

$$\left.\begin{aligned} &\int_{(S^{(l)})}(\varrho_{2n} - \varrho_{2n-1})\,d\sigma = 2\int_{(S^{(l)})} f\,d\sigma, \\ &\int_{(S^{(l)})}(A - B)\,d\sigma = \int_{(S^{(l)})}\psi\,d\sigma = 2\int_{(S^{(l)})} f\,d\sigma. \end{aligned}\right| \tag{81}$$

From the last equation it follows that ψ is different from zero if one of the conditions (69) is not satisfied.

We now introduce k functions $f^{(1)}, f^{(2)}, \ldots, f^{(k)}$ defined in the following manner:

$$f^{(l)} = 0, \text{ for } M_0 \text{ not on } (S^{(l)})\ ;$$

$$f^{(l)} = \frac{1}{2\,S^{(l)}}, \text{ for } M_0 \text{ on } (S^{(l)})\ .$$

We form the corresponding functions

$$A^{(1)}, A^{(2)}, \ldots, A^{(k)}, \qquad B^{(1)}, B^{(2)}, \ldots, B^{(k)};.$$

we obtain the corresponding eigenfunctions

$$\psi^{(1)}, \psi^{(2)}, \ldots, \psi^{(k)} \tag{82}$$

by putting

$$\psi^{(l)} = A^{(l)} - B^{(l)}.$$

From (81)

$$\int_{(S^{(\lambda)})}\psi^{(l)}\,d\sigma = 0 \quad \text{for} \quad \lambda \neq l, \quad \int_{(S^{(l)})}\psi^{(l)}\,d\sigma = 1, \quad \int_{(S^{(l)})} f^{(l)}\,d\sigma = \tfrac{1}{2}. \tag{83}$$

The eigenfunctions (82) are linearly independent. Indeed, from (83) we can conclude that the identity

$$C_1\psi^{(1)} + C_2\psi^{(2)} + \cdots + C_k\psi^{(k)} \equiv 0,$$

where $C_1, C_2, \ldots, C_k$ are some constants, can hold only if all the C_l vanish.

It is easily verified that the eigenfunction corresponding to an arbitrary function f is a linear combination of the functions (82); it is equal to the sum

$$\alpha^{(1)}\psi^{(1)} + \alpha^{(2)}\psi^{(2)} + \cdots + \alpha^{(k)}\psi^{(k)}$$

with

$$\int_{(S^{(l)})} f\, d\sigma = \tfrac{1}{2}\alpha^{(l)}.$$

Indeed, if ψ is the eigenfunction corresponding to f, then the function

$$f' = f - \alpha^{(1)} f^{(1)} - \cdots - \alpha^{(k)} f^{(k)} \tag{84}$$

gives rise to the eigenfunction

$$\psi - \alpha^{(1)}\psi^{(1)} - \cdots - \alpha^{(k)}\psi^{(k)}.$$

However, since f' satisfies the conditions (69), the last function is identically zero.

It is easily seen that one can obtain all the solutions of the homogeneous equation for $\zeta = -1$ by the method presented. If ψ is such a solution one can put $f = \frac{1}{2}\psi$. Since

$$\psi(0) = \frac{1}{2\pi}\int_{(S_1)} \psi(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\, d\sigma_1$$

it then follows that

$$\varrho_0 = \tfrac{1}{2}\psi, \quad \varrho_1 = -\tfrac{1}{2}\psi, \quad \varrho_2 = \tfrac{1}{2}\psi, \ldots,$$

the procedure thus leads back to the function ψ.

From the foregoing considerations we can conclude that the function ψ is completely determined by the values of the integrals

$$\int_{(S^{(l)})} \psi\, d\sigma \qquad (l = 1, 2, \ldots, k).$$

§18. A Remark on the Question of Whether the Solution of the NEUMANN Problem Belongs to the Class $H(l,A,\lambda)$

As we have seen, each solution of the NEUMANN problem can be represented as the potential of a simple layer $V[\mu]$; the density μ is obtained as the solution of the integral equation

$$\mu(0) = -\frac{\zeta}{2\pi}\int_{(S_1)} \mu(1)\frac{\cos(r_{10}N_0)}{r_{10}^2}\, d\sigma_1 + \frac{f(0)}{2\pi}, \tag{85}$$

which we shall write in the form

$$\mu(0) = -\frac{\zeta}{2\pi}\frac{dV[\mu]}{dn} + \frac{f(0)}{2\pi}. \tag{86}$$

For $\zeta = 1$ $V[\mu]$ is a solution of the inner problem $\frac{dV_i}{dn} = f$, and for $\zeta = -1$ $V[\mu]$ is a solution of the outer problem $\frac{dV_e}{dn} = -f$.

To have a specific case in mind, we consider the region of the ordinary case and investigate whether the solution of the NEUMANN problem belongs to the class $H(l,A,\lambda)$ in dependence on whether the surface belongs to the class $L_k(B,\lambda)$ and the function f to the class $H(l,A,\lambda)$ on (S). In the ordinary case equation (2) ($\zeta = -1$) for an arbitrary continuous function $f(0)$ has a unique solution. The resolvent of the kernel $K_n(1,0)$ at $\zeta = -1$, which we shall denote by $R_n(1,0)$, is bounded in the ordinary case. Hence,

$$\begin{aligned} |\mu(0)| &\leq |\textstyle\sum_n(0)| + \left|\int\limits_{(S_1)} R_n(1,0)\textstyle\sum_n(1)\,d\sigma_1\right| \\ &\leq \mathrm{Max}\,|\textstyle\sum_n|\left(1 + \mathrm{Max}\int\limits_{(S_1)}|R_n(1,0)|\,d\sigma_1\right) \\ &\leq C\cdot\mathrm{Max}\,|f|; \end{aligned}$$

here it is clear that $\mathrm{Max}\,|\Sigma_n| \leqq C_1\,\mathrm{Max}\,|f|$. Therefore, if A is an arbitrary upper bound for f

$$|\mu| \leq CA. \tag{87}$$

If $\zeta = 1$ then equation (86) has a solution only if f satisfies the condition

$$\int\limits_{(S)} f\,d\sigma = 0. \tag{88}$$

If condition (88) holds, then equation (86) has multiple solutions which differ by summands of the form $c\varrho$ where ϱ is the ROBIN function. Among these solutions there is one which satisfies condition (88). We shall show that the solution of equation (86) thus chosen also satisfies inequality (87). Indeed, from the theory of integral equations the resolvent of the kernel $K_n(1,0)$ in a neighborhood of the point $\zeta = 1$ has the representation

$$\frac{\varrho(M_0)}{\zeta - 1} + B(\zeta;\,M_1,\,M_0),$$

where $B(\zeta;M_1,M_0)$ is bounded in a neighborhood of the point $\zeta = 1$. If condition (88) is satisfied, then from this it follows that

$$\begin{aligned}\mu(\zeta;M_0) &= \sum\nolimits_n(\zeta;M_0)+\zeta^n\int\limits_{(S_1)}\left[\frac{\varrho(M_0)}{\zeta-1}+B(\zeta;M_1,M_0)\right]\sum\nolimits_n(\zeta;M_1)\,d\sigma_1\\ &= \sum\nolimits_n(\zeta;M_0)+\zeta^n\int\limits_{(S_1)}B(\zeta;M_1,M_0)\sum\nolimits_n(\zeta;M_1)\,d\sigma_1,\end{aligned}\tag{89}$$

where Σ_n with f also satisfies condition (88). From this we conclude that $\mu(\zeta;M_0)$ for all ζ sufficiently close to $\zeta = 1$ satisfies inequality (87). Moreover, integrating both sides of equation (85) over (S_0), we find:

$$\int\limits_{(S_0)}\mu(0)\,d\sigma_0=\frac{1}{2\pi(1-\zeta)}\int\limits_{(S_0)}f(0)\,d\sigma_0;$$

hence, the solution (89) for $\zeta\neq 1$ satisfies condition (88) if this is the case for f. The function $\mu(\zeta;M_0)$ is an analytic function of the parameter ζ in a neighborhood of the point $\zeta = 1$ and therefore also satisfies condition (88) at $\zeta = 1$. In the following we shall treat the cases $\zeta = -1$ and $\zeta = 1$ together; in the case $\zeta = 1$ we shall understand by μ the solution mentioned above, while we shall assume without reservation that f satisfies condition (88).

The solution μ of equation (86) for $\zeta = 1$ thus chosen yields that solution of the inner problem which satisfies the condition

$$\int\limits_{(S)}\varrho V\,d\sigma=0\,.$$

It is this solution which will be considered.

Lemma. *If* $(S)\in L_{k+1}(B,\lambda)$ $(k\geqq 0)$ *and* $f\in H(l,A,\lambda)$ *on* (S) $(0\leqq l\leqq k)$, *then* $\mu\in H(l,cA,\lambda')$ *on* (S).

Proof. From (87) $|\mu|<cA$. From Theorem 2 of II, §21 (H-continuity of the normal derivative of the potential of a simple layer with bounded density) it then follows that

$$\frac{dV[\mu]}{dn}\in H(0,c_1A,\lambda')\,.$$

Since $f\in H(l,A,\lambda)$, it follows that $f\in H(0,A\sqrt{2},\lambda)$ and thus since μ satisfies equation (86) that

$$\mu\in H(0,c_2A,\lambda')\quad\text{on}\ |(S)\,.$$

We shall assume that $\mu\in H(l',c''A,\lambda'')$ on (S) with

$$0\leq l'\leq l-1,\ \lambda''<\lambda,$$

and now show that the relation $\mu\in H(l'+1,c'A,\lambda')$ with $\lambda'<\lambda''$ holds. Indeed, $l'+2\leqq l+1\leqq k+1$ and therefore $(S)\in L_{l'+2}$. Using Theorem 4 of II, §21, it follows from $\mu\in H(l',c''A,\lambda'')$ that

$$\frac{dV[\mu]}{dn}\in H(l'+1,c_1'A,\lambda')\qquad(\lambda'<\lambda'')\,.$$

Since $f\in H(l'+1,A,\lambda)$, it follows from equation (86) that

$$\mu \in H(l' + 1, c_2' A, \lambda'). \tag{90}$$

It has already been shown that the function μ belongs to the class $H(0,c_2 A,\lambda')$; it follows therefore from (90) that $\mu \in H(1,c_3 A,\lambda'), \ldots, \mu \in H(l,c_{l+2} A,\lambda')$. The lemma has now been proved.

Remark. Let $f \equiv 0$. If the homogeneous equation (86) in this case has a solution ϱ with $|\varrho| < A_0$, then clearly

$$\varrho \in H(k, cA_0, \lambda').$$

If $f \in H(l,A,\lambda)$ $(0 \leqq l \leqq k)$, then it follows from the lemma just proved and Theorem 4 of II, §19 that $V[\mu] \in H(l + 1,cA,\lambda')$. This gives the theorem:

Theorem 1. *If* $(S) \in L_{k+1}(B,\lambda)$ *and* $f \in H(l,A,\lambda)$ $(k \geqq 0, 0 \leqq l \leqq k)$, *then the solution of the inner (outer)* NEUMANN *problem belongs to the class* $H(l + 1,cA,\lambda')$ *in* (D_i) *(in* (D_e)*).*

Remark. Let ϱ_1 be that solution of equation (86) for $f \equiv 0$ and $\zeta = 1$ for which $V[\mu]$ is equal to one on (S) and in (D_i). Further let Max $\varrho_1 = A_0$. As already mentioned, $\varrho_1 \in H(k,cA_0,\lambda')$, and hence

$$V[\varrho_1] \in H(k+1, c_1 A_0, \lambda').$$

This remark will be of use in studying the solution of the outer DIRICHLET problem (IV, §18).

From Theorem 1 there follows an analogous theorem for surfaces and functions f of class $C^{(k)}$.

Theorem 2. *If* $(S) \in C^{(k+1)}(B)$ $(k \geqq 1)$ *and* $f \in C^{(l)}(A)$ $(0 \leqq l \leqq k)$, *then the solution of the inner (outer)* NEUMANN *problem in* (D_i) *(in* (D_e)*) belongs to the class* $C^{(l)}(cA)$.

Proof. If $l \geqq 1$,

$$f \in H(l-1, cA, 1).$$

Moreover, $(S) \in L_k(c_1 B,1)$; from Theorem 1 the solution of the NEUMANN problem belongs to the class $H(l,c_2 A,\lambda)$ $(\lambda < 1)$ and hence to the class $C^{(l)}(c_2 A)$. If $l = 0$, i.e:, if f is a continuous function, then μ is also continuous and satisfies inequality (87). The continuity of the solution of the NEUMANN problem and the validity of the inequality

$$|V| < cA$$

are then obvious. This completes the proof of Theorem 2.

§19. On the Uniqueness of the Solution of the NEUMANN Problem[1]

We shall treat the inner problem in the ordinary case and in case (J) and the outer problem in the ordinary case and in case (E). In these cases the solution in a connected region will be sought. We shall not distinguish these cases and shall denote the region in question by (D).

Theorem 1. *A function harmonic in the interior of (D) is equal to a constant (equal to zero in the case of the outer problem) if its normal derivative vanishes.*

Let $T(\alpha,k,h)$ be the solid of revolution bounded by the surface $z = k(x^2 + y^2)^{\frac{1+\alpha}{2}}$ $(k > 0, \alpha > 0)$ and the plane $z = h$. We shall call the point $(0,0,0)$ the apex of the solid and the part of the boundary lying in the plane $z = h$ the base.

We shall say that the connected region (D) belongs to the class A if to each boundary point M of this region there can be assigned a solid T' congruent to the solid $T(\alpha,k,h)$ having its apex at M and contained in $(D + S)$. If the numbers α, k, h are independent of the point M we shall say that (D) belongs to the class B.

One easily verifies that regions bounded by LYAPUNOV surfaces belong to the class B. Indeed, taking the surface point M as the origin of our coordinate system and the corresponding tangent plane as the (x,y) plane, then for the subregion of the surface contained in a LYAPUNOV sphere about M:

$$|z| < b\,(x^2 + y^2)^{\frac{1+\lambda}{2}}.$$

If we choose $k > b$ and $\alpha < \lambda$, then the portion of the lateral surface of the solid of revolution in a neighborhood of the apex lies in (D). The quantity h is now chosen such that the entire solid $T(\alpha,k,h)$ lies inside the LYAPUNOV sphere about M. Since b, λ, and the radius of the LYAPUNOV sphere are independent of the location of the point M, it is possible to choose the numbers k, α, and h independent of M.

Theorem 1 is a consequence of the following more general theorem which we shall now prove.

Theorem 2. *Let U be a nonconstant function harmonic in the interior of the region (D) and M_0 be a boundary point of (D) at which this function has the boundary value U_0 which is equal to the lower bound of its values in (D).*

If a solid T' can be enclosed in the region (D) which is congruent to a solid $T(\alpha,k,h)$ and has its apex at M_0, then

[1] In this section we present the work contained in the paper by M. V. KELDYSH and M. A. LAVRENT'EV, Doklady AN SSSR, Vol. XVI, No. 3, 1937.

$$\lim_{r_{10}\to 0} \frac{U(M_1)-U_0}{r_{10}} > 0, \tag{91}$$

where M_1 is a point on the axis of the solid $T(\alpha,k,h)$.

Proof. Let U_1 be the minimum of U on the base of the solid $T(\alpha,k,h)$. Clearly, $U_1 > U_0$, since if $U_1 = U_0$ this would mean that the harmonic function U assumed its minimum in the interior of the region (D).

We shall now assume that there exists a harmonic function W in $T(\alpha,k,h)$ such that on the base

$$W < U_1, \tag{92}$$

on the lateral surface

$$W \leq U_0, \tag{93}$$

at the apex of the solid

$$W = U_0,$$

and furthermore W satisfies the inequality

$$\lim_{r_{10}\to 0} \frac{W(M_1)-W(M_0)}{r_{10}} > 0 \tag{94}$$

From the maximum principle for harmonic functions this would imply that

$$W \leq U \qquad \text{in } T(\alpha,k,h)$$

and hence

$$\frac{U(M_1)-U_0}{r_{10}} \geq \frac{W(M_1)-W(M_0)}{r_{10}}. \tag{95}$$

From (94) and (95) then would follow that for $r_{10} \to 0$ the inequality (91), so that the proof of Theorem 2 would be complete. It therefore remains to establish the existence of a harmonic function W satisfying conditions (92), (93), and (94) having the value U_0 at the apex of the solid $T(\alpha,k,h)$.

Let α and k be fixed. We shall subsequently determine the value of h. We put

$$W = \gamma\,[r\cos\Theta + r^{1+\beta}P_{1+\beta}(\cos\Theta)] + U_0,$$

where γ and β are positive constants and $r = \sqrt{x^2+y^2+z^2}$; Θ denotes the angle between the radius vector of the point (x,y,z) and the z axis; $P_{1+\beta}(t)$ is the solution of the LEGENDRE differential equation of order $1+\beta$ which is regular at $t = 1$ and assumes there the value one:

$$\frac{d}{dt}\left[(1-t^2)\frac{dP}{dt}\right] + (1+\beta)(2+\beta)P(t) = 0.$$

Clearly, $W(M_0) = U_0$. One easily checks that W is a harmonic function. Moreover, on the axis of the solid $\cos\Theta = 1$, and we obtain:

$$\frac{W(M_1)-U_0}{r_{10}} = \gamma(1+r_{10}^{\beta}) \to \gamma > 0 \qquad (r_{10}\to 0),$$

i.e., condition (94) is fulfilled.

We shall now show that by appropriate choice of γ and β conditions (92) and (93) can also be satisfied.

Indeed, in the interval $-1 < t \leqq 3P_{1+\beta}(t)$ has the series representation

$$P_{1+\beta}(t)$$
$$= 1 + \sum_{k=1}^{\infty} \frac{2+3\beta+\beta^2}{2\cdot 1^2} \cdot \frac{2+3\beta+\beta^2-2}{2\cdot 2^2} \cdots \frac{2+3\beta+\beta^2-k(k-1)}{2\cdot k^2} (t-1)^k \text{ [1]);}$$

hence,

$$\begin{aligned} P_{1+\beta}(0) = 1 &- \frac{2+3\beta+\beta^2}{2\cdot 1^2} + \frac{2+3\beta+\beta^2}{2\cdot 1^2} \cdot \frac{3\beta+\beta^2}{2\cdot 2^2} \\ &+ \frac{2+3\beta+\beta^2}{2\cdot 1^2} \cdot \frac{3\beta+\beta^2}{2\cdot 2^2} \cdot \left[\frac{4-3\beta-\beta^2}{2\cdot 3^2}\right. \\ &+ \frac{4-3\beta-\beta^2}{2\cdot 3^2} \cdot \frac{10-3\beta-\beta^2}{2\cdot 4^2} + \cdots \\ &+ \left.\frac{4-3\beta-\beta^2}{2\cdot 3^2} \cdot \frac{10-3\beta-\beta^2}{2\cdot 4^2} \cdots \frac{m^2+3m-3\beta-\beta^2}{2(m+2)^2} + \cdots\right]. \end{aligned}$$

If $|\beta| < 1$, then the series in the square brackets is positive and less than one, since for $m \geqq 1$

$$0 < \frac{m^2+3m-3\beta-\beta^2}{2(m+2)^2} = \frac{m^2+4m+4}{2(m+2)^2} - \frac{m+4+3\beta+\beta^2}{2(m+2)^2} < \frac{1}{2}.$$

Therefore,

$$\begin{aligned} P_{1+\beta}(0) &< -\frac{3\beta+\beta^2}{2} + \frac{(3\beta+\beta^2)(2+3\beta+\beta^2)}{8} \\ &= -\frac{3\beta+\beta^2}{4}\left(1-\frac{3\beta+\beta^2}{2}\right). \end{aligned}$$

The positive constant β is now chosen such that the two conditions

$$0 < \frac{3\beta+\beta^2}{2} < 1 \quad \text{and} \quad \beta < \alpha$$

are satisfied.

From the first condition it follows that

$$P_{1+\beta}(0) < 0.$$

We shall now study the behavior of the function

$$r\cos\Theta + r^{1+\beta} P_{1+\beta}(\cos\Theta)$$

on the lateral surface of the solid $T(\alpha,k,h)$. We have:

$$r\cos\Theta = z = k(x^2+y^2)^{\frac{1+\alpha}{2}} = k(r\sin\Theta)^{1+\alpha} = k r^{1+\alpha}\sin^{1+\alpha}\Theta$$

and hence

$$\begin{aligned} r\cos\Theta + r^{1+\beta}P_{1+\beta}(\cos\Theta) &= k r^{1+\alpha}\sin^{1+\alpha}\Theta + r^{1+\beta}P_{1+\beta}(\cos\Theta) \\ &= r^{1+\beta}\left[k r^{\alpha-\beta}\sin^{1+\alpha}\Theta + P_{1+\beta}(\cos\Theta)\right]. \end{aligned}$$

If the point of the lateral surface tends toward the apex, then $r \to 0$ and

$\Theta \rightarrow \frac{\pi}{2}$; hence, the quantity in the square brackets remains negative for all Θ of a certain interval

$$\Theta_0 < \Theta < \frac{\pi}{2} \qquad (\Theta_0 > 0),$$

We choose h so small that on the lateral surface of the solid $T(\alpha,k,h)$ the inequality $\Theta > \Theta_0$ holds. Condition (93) is then satisfied for arbitrary $\gamma > 0$ on the lateral surface of this solid $T(\alpha,k,h)$.

Since $U_0 < U_1$, through choice of a sufficiently small $\gamma > 0$ one can also achieve that inequality (92) is satisfied. Theorem 2 is herewith proved. It affords the following conclusion: If the normal derivative of a nonconstant function harmonic in the interior of (D) exists at each boundary point of the region (D), then there exists at least one boundary point at which this derivative is different from zero. This then proves Theorem 1.

CHAPTER IV

THE DIRICHLET PROBLEM

§1. The Statement of the DIRICHLET Problem

Problem A. *It is required to find a function V harmonic in the interior of (D_i) which satisfies the condition*

$$V_i = f \quad \text{on} \quad (S) \tag{1}$$

where f is a given function, defined and continuous on the boundary.

This is the inner DIRICHLET problem. The outer DIRICHLET problem is similar:

It is required to find a function V harmonic in the interior of (D_e) which satisfies the condition

$$V_e = f \quad \text{on} \quad (S) \tag{2}$$

where f is a given function, defined and continuous on the boundary.

Depending on the character of the region (D_i), one treats the ordinary case, the case (J), or the case (E). It is assumed that (D_i) is bounded by surfaces which satisfy the LYAPUNOV conditions.

In case (J) the outer problem is equivalent to an outer and several inner problems of the ordinary case. In case (E) the inner DIRICHLET problem can be replaced by several inner problems of the ordinary case.

One easily verifies that the DIRICHLET problem can have no more than one solution. Indeed, if U and V are two functions harmonic in (D_i) which coincide on (S), then their difference $U - V$ is harmonic in the interior of (D_i) and vanishes on (S). As we have seen in I, §8 a function harmonic in the interior of (D_i) assumes its maximum and minimum only on the boundary of the region. Therefore, the maximum and minimum of the difference $U - V$ are zero, and hence $U - V \equiv 0$ in (D_i), i.e., $U \equiv V$. This establishes the uniqueness of the solution of the DIRICHLET problem.

Let the function V be defined in (D_e) and let $V(R)$ be the maximum of its absolute value on the sphere of radius R about a fixed point of space. We shall say that V is zero at infinity if $V(R)$ tends to the limit zero as $R \to \infty$.

If two functions U and V possess continuous second derivatives in (D_e), if at each interior point they satisfy the LAPLACE equation, if they are further zero at infinity, and if in addition they assume the same values on (S), then from the same principle of the maximum U and V coincide in (D_e). We shall see in what follows that the solution of the outer DIRICHLET problem always exists and is given either by the potential of a double layer alone or by the potential of a double and a single layer. These potentials have moreover the property that their product with the first power of R and the product of their first derivatives with the second power of R remain bounded as $R \to \infty$.

Every function v which satisfies the LAPLACE *equation at every interior point of (D_e) and vanishes at infinity has the property that the products*

$$R v, \qquad R^2 \frac{\partial v}{\partial x}, \qquad R^2 \frac{\partial v}{\partial y}, \qquad R^2 \frac{\partial v}{\partial z}$$

remain bounded as $R \to \infty$.

To prove this, let (D_e') be the subregion of (D_e) which lies outside the sphere Σ_R, the radius R of which is so large that the entire boundary of the region (D_e) lies inside Σ_R. Let U be the solution of the outer DIRICHLET problem for (D_e') which assumes the value $v(S)$ on the outer surface (S) of Σ_R. Then v and U agree in (D_e'), and hence in (D_e') v is the sum of the potentials of a simple layer and double layer on (S). This now establishes our assertion with regard to the behavior of the function v and its first derivatives as $R \to \infty$. This also proves the assertion in footnote 7 of Chapter I.

§2. Replacing Problem A by Another Problem

In place of Problem A we shall treat the following problem:

Problem B. *It is required to find the potential of a double layer on (S) which satisfies condition* (1) *for the inner problem and condition* (2) *for the outer problem.*

We shall see that Problems A and B are not equivalent; there are cases in which Problem B has no solution. The study of Problem B leads us to the complete solution of Problem A.

We shall now replace Problem B by the more general Problem C.

Problem C. *It is required to find the potential W of a double layer which satisfies the equation*

$$W_i - W_e = 2\zeta \overline{W} + 2f \quad \text{on} \quad (S) \tag{C}$$

The bar over a letter always indicates that the value of the corresponding function at a point of (S) is meant.

W is a function of ζ; one sees easily that for $\zeta = 1$ this function is the

solution of the outer Problem B and for $\zeta = -1$ the solution of the inner Problem B. Indeed, according to the formulas of II, §3,

$$W_i - W_e = 4\pi \times \text{density of } W, \quad W_i + W_e = 2\overline{W}. \tag{3}$$

With the help of these formulas we find from equation (C) for $\zeta = 1$:

$$W_i - W_e = W_i + W_e + 2f, \quad W_e = -f.^{1)}$$

For $\zeta = -1$ we obtain from the same equation:

$$W_i - W_e = -W_i - W_e + 2f, \quad W_i = f.$$

This proves our assertion.

We denote the density of the potential W by $\frac{1}{2\pi}\vartheta$ or—what is the same—we put

$$W = \frac{1}{2\pi}\int\limits_{(S_1)} \vartheta(1)\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1. \tag{4}$$

From the first equation in (3) it follows that

$$W_i - W_e = 4\pi\frac{\vartheta(0)}{2\pi} = 2\vartheta.$$

Substituting this into (C), we find:

$$\vartheta(0) = \frac{\zeta}{2\pi}\int\limits_{(S_1)} \vartheta(1)\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1 + f(0). \tag{5}$$

The problem is thus reduced to the study of the integral equation (5).

§3. The Formal Solution of Problem C

We wish to find the solution of Problem C in the form of a series, and we put

$$\vartheta = \vartheta_0 + \vartheta_1\zeta + \vartheta_2\zeta^2 + \cdots. \tag{6}$$

Substituting (6) into (5) and comparing like powers of ζ, we obtain

$$\left.\begin{aligned}
&\vartheta_0 = f,\\
&\vartheta_1 = \frac{1}{2\pi}\int\limits_{(S_1)} \vartheta_0(1)\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots,\\
&\vartheta_n = \frac{1}{2\pi}\int\limits_{(S_1)} \vartheta_{n-1}(1)\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1,\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right\} \tag{7}$$

If we assume that the series (6) converges uniformly on (S) for certain values of ζ, we find on substituting (6) into (4) that

$$W = W_1 + W_2\zeta + W_3\zeta^2 + \cdots + W_n\zeta^{n-1} + \cdots \tag{8}$$

with

$$\left.\begin{aligned} W_1 &= \frac{1}{2\pi}\int\limits_{(S_1)} \vartheta_0(1)\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1, \\ &\dots\dots\dots\dots\dots\dots, \\ W_n &= \frac{1}{2\pi}\int\limits_{(S_1)} \vartheta_{n-1}(1)\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1, \\ &\dots\dots\dots\dots\dots\dots. \end{aligned}\right\} \tag{9}$$

One can write the equations (9) in another form which makes it possible to compute the potentials W_k in succession without making use of the functions (7).

If we substitute (8) into equation (C), we find:

$$\begin{aligned} (W_1)_i - (W_1)_e &= 2f, \\ (W_2)_i - (W_2)_e &= 2\overline{W}_1, \\ &\dots\dots\dots, \\ (W_n)_i - (W_n)_e &= 2\overline{W}_{n-1}. \\ &\dots\dots\dots. \end{aligned}$$

With the help of the first formula in (3) we obtain from these equations the relations

$$2f = 4\pi\frac{\vartheta_0(0)}{2\pi}, \quad 2\overline{W}_1 = 4\pi\frac{\vartheta_1(0)}{2\pi}, \quad \dots, \quad 2\overline{W}_{n-1} = 4\pi\frac{\vartheta_{n-1}(0)}{2\pi}, \quad \dots.$$

From this it follows finally that

$$f = \vartheta_0(0), \quad \overline{W}_1 = \vartheta_1(0), \quad \dots, \quad \overline{W}_{n-1} = \vartheta_{n-1}(0), \quad \dots. \tag{10}$$

From formulas (10) it follows that equations (9) can be replaced by the equations

$$\left.\begin{aligned} W_1 &= \frac{1}{2\pi}\int\limits_{(S_1)} f(1)\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1, \\ W_2 &= \frac{1}{2\pi}\int\limits_{(S_1)} \overline{W}_1\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1, \\ &\dots\dots\dots\dots\dots\dots, \\ W_n &= \frac{1}{2\pi}\int\limits_{(S_1)} \overline{W}_{n-1}\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1, \\ &\dots\dots\dots\dots\dots\dots, \end{aligned}\right\} \tag{9'}$$

these equations clearly have the property mentioned above. The potentials (9′) are called NEUMANN *potentials.*

We thus see that the solution of Problem B reduces to the question of whether the series (8) converges at $\zeta = 1$ and $\zeta = -1$.

§4. The Actual Solution of Problem C

The kernel of equation (5) reads

$$\overline{K}(1, 0) = \frac{1}{2\pi} \frac{\cos(r_{10} N_1)}{r_{10}^2}. \tag{11}$$

In solving the NEUMANN problem we studied the equation

$$\mu(0) = -\frac{\zeta}{2\pi} \int_{(S_1)} \mu(1) \frac{\cos(r_{10} N_0)}{r_{10}^2} d\sigma_1 + f(0) \tag{12}$$

with the kernel

$$K(1, 0) = -\frac{1}{2\pi} \frac{\cos(r_{10} N_0)}{r_{10}^2}. \tag{13}$$

$\overline{K}(1,0)$ and $K(1,0)$ are associated kernels. Indeed, we find on interchanging the indices 1 and 0 in (13) that

$$K(0, 1) = -\frac{1}{2\pi} \frac{\cos(r_{01} N_1)}{r_{01}^2} = \frac{1}{2\pi} \frac{\cos(r_{10} N_1)}{r_{10}^2} = \overline{K}(1, 0).$$

From this it follows that one can obtain any iterated kernel $\overline{K}_n(1,0)$ from the corresponding kernel $K_n(1,0)$ by interchanging the indices 1 and 0. Thus, for example,

$$\begin{aligned} \overline{K}_2(1, 0) &= \int_{(S_2)} \overline{K}(1, 2)\, \overline{K}(2, 0)\, d\sigma_2 \\ &= \int_{(S_2)} K(2, 1)\, K(0, 2)\, d\sigma_2 = \int_{(S_2)} K(0, 2)\, K(2, 1)\, d\sigma_2 = K_2(0, 1). \end{aligned}$$

From the theorem in III, §4 we know that the kernel $\overline{K}_n(1,0) = K_n(0,1)$ is bounded if n satisfies the inequality

$$2 - n\lambda < 0.$$

Thus, writing equation (5) in the form

$$\vartheta(0) = \zeta \int_{(S_1)} K(0, 1)\, \vartheta(1)\, d\sigma_1 + f(0), \tag{5'}$$

we can reduce the determination of $\vartheta(0)$ to finding the solutions of the equation

$$\vartheta(0) = \zeta^n \int_{(S_1)} K_n(0, 1)\, \vartheta(1)\, d\sigma_1 + C_n(0) \tag{14}$$

wherein $C_n(0)$ denotes the sum of the first n terms of series (6). We have

similarly in III, §5 reduced the solving of equation (12) to solving the equation

$$\mu(0) = \zeta^n \int_{(S_1)} K_n(1, 0)\, \mu(1)\, d\sigma_1 + \textstyle\sum_n(0)\,. \tag{14'}$$

The associated equations (14) and (14′) have the common resolvent

$$\frac{D(\zeta, 1, 0)}{D(\zeta)} \tag{15}$$

which was introduced in III, §5; in this section we obtained:

$$\mu(0) = \textstyle\sum_n(0) + \zeta^n \displaystyle\int_{(S_1)} \textstyle\sum_n(1) \dfrac{D(\zeta, 1, 0)}{D(\zeta)}\, d\sigma_1 = \dfrac{D_1(\zeta, 0)}{D_1(\zeta)}\,. \tag{16}$$

In the case at hand we find:

$$\vartheta(0) = C_n(0) + \zeta^n \int_{(S_1)} C_n(1) \frac{D(\zeta, 0, 1)}{D(\zeta)}\, d\sigma_1 = \frac{D_2(\zeta, 0)}{D_2(\zeta)}\,. \tag{17}$$

The solution found now leads to the solution of Problem B if the mermorphic function (17) does not have poles at $\zeta = 1$ and $\zeta = -1$. We shall now establish under what conditions this is the case.

In the previous section when studying the function (16) we proved:

(α) function (16) has no nonreal poles;

(β) there are no poles between -1 and $+1$;

(γ) if $\zeta = 1$ and $\zeta = -1$ are poles of the function (16), then these poles are simple.

Out investigation would be considerably simpler if we could show that the function (17) also had these properties. This is the case if the following assertions (α), (β), (γ) hold; we do not here consider poles of the fraction (17) the absolute value of which is greater than one, since these are of no importance for the problem at hand.

(α) the fraction (15) has no complex poles with absolute value less than one;

(β) except for possibly $\zeta = 1$ and $\zeta = -1$, the fraction (15) has no poles of absolute value equal to one;

(γ) if $\zeta = 1$ and $\zeta = -1$ are poles of the fraction (15), then these poles are simple.

The denominator of the fraction (17) is obtained from the denominator of fraction (15) by cancellation; if assertions (α), (β), and (γ) hold for (15), then they also hold for (17).

Remark. The proof that the fraction (17) has these and even more general properties can be carried through with the methods used in III, §7. In this case, however, one must make use of the normal derivatives of the potential of the double layer. These derivatives exist only when certain additional conditions are fulfilled, so that we should first of all have to establish that the potential W satisfies these conditions.

§5. Some Remarks on the Kernel $K_n(1,0)$ [1]

In the following investigation of the kernel $K_n(1.0)$ it will always be assumed that n is odd. We first of all note three things:

1. If the point M_1 lies on (S), then in all cases (in the ordinary case and in cases (J) and (E))

$$\int_{(S)} K_n(1,0)\,d\sigma = 1. \tag{18}$$

Indeed,

$$\int_{(S)} K_n(1,0)\,d\sigma = \int_{(S)}\left(\int_{(S_2)} K_{n-1}(1,2)\,K(2,0)\,d\sigma_2\right)d\sigma$$
$$= \int_{(S_2)} K_{n-1}(1,2)\left(\int_{(S)} K(2,0)\,d\sigma\right)d\sigma_2.$$

But since

$$\int_{(S)} K(2,0)\,d\sigma = -\frac{1}{2\pi}\int_{(S)}\frac{\cos(r_{20}N_0)}{r_{20}^2}\,d\sigma = \frac{1}{2\pi}\int_{(S)}\frac{\cos(r_{02}N_0)}{r_{20}^2}\,d\sigma = 1,$$

it follows for a point M_1 on (S):

$$\int_{(S)} K_n(1,0)\,d\sigma = \int_{(S_2)} K_{n-1}(1,2)\,d\sigma_2 = \int_{(S)} K_{n-1}(1,0)\,d\sigma$$

and

$$\int_{(S)} K_n(1,0)\,d\sigma = \int_{(S)} K_{n-1}(1,0)\,d\sigma = \cdots = \int_{(S)} K(1,0)\,d\sigma = 1.$$

2. In case (E) for a point M_1 on (S)

$$\left.\begin{aligned} &\int_{(S^{(l)})} K_n(1,0)\,d\sigma = 1, \quad \text{for } M_1 \text{ on } (S^{(l)}), \\ \text{and} \quad &\int_{(S^{(l)})} K_n(1,0)\,d\sigma = 0, \quad \text{for } M_1 \text{ not on } (S^{(l)}). \end{aligned}\right\} \tag{19}$$

Indeed, in this case

$$\int_{(S^{(l)})} K_n(1,0)\,d\sigma = \int_{(S_2)} K_{n-1}(1,2)\left(\int_{(S^{(l)})} K(2,0)\,d\sigma\right)d\sigma_2$$
$$= \int_{(S_2^{(l)})} K_{n-1}(1,2)\,d\sigma_2,$$

for the potential of the double layer

$$\int_{(S^{(l)})} K(2,0)\,d\sigma \tag{20}$$

is zero if M_2 is not on $(S^{(l)})$ and equals one if M_2 lies on $(S^{(l)})$.

From this it now follows that

$$\int\limits_{(S^{(l)})} K_n(1, 0)\, d\sigma = \int\limits_{(S^{(l)})} K_{n-1}(1, 0)\, d\sigma = \cdots = \int\limits_{(S^{(l)})} K(1, 0)\, d\sigma .$$

If the point M_1 does not lie on $(S^{(l)})$, then the last integral is equal to zero; otherwise it is equal to one.

3. In case (J) for a point M_1 on (S)

$$\left.\begin{aligned} &\int\limits_{(S^{(l)})} K_n(1, 0)\, d\sigma = (-1)^n, && \text{for } M_1 \text{ on } (S^{(l)}), \\ \text{and} \quad &\int\limits_{(S^{(l)})} K_n(1, 0)\, d\sigma = 0, && \text{for } M_1 \text{ not on } (S^{(l)})\,; \end{aligned}\right\} \tag{21}$$

here $(S^{(l)})$ denotes an inner boundary surface.

In this case

$$\begin{aligned} \int\limits_{(S^{(l)})} K_n(1, 0)\, d\sigma &= \int\limits_{(S_2)} K_{n-1}(1, 2) \Bigg(\int\limits_{(S^{(l)})} K(2, 0)\, d\sigma \Bigg) d\sigma_2 \\ &= (-1) \int\limits_{(S_2^{(l)})} K_{n-1}(1, 2)\, d\sigma_2 ; \end{aligned}$$

the potential (20) is here equal to zero as long as M_2 does not lie on $(S^{(l)})$ and equals -1 if M_2 lies on $(S^{(l)})$, since the normal is now directed into the interior of $(S^{(l)})$. From this it follows that

$$\int\limits_{(S^{(l)})} K_n(1, 0)\, d\sigma = (-1) \int\limits_{(S^{(l)})} K_{n-1}(1, 0)\, d\sigma = \cdots = (-1)^{n-1} \int\limits_{(S^{(l)})} K(1, 0)\, d\sigma .$$

If M_1 does not lie on $(S^{(l)})$, then the last integral is equal to zero; for a point on $(S^{(l)})$ it has the value -1.

§6. Proof of the Assertions Made in §4

In III, §3 we obtained by solving equation (12) the series

$$\mu(0) = \varrho_0(0) + \zeta \varrho_1(0) + \zeta^2 \varrho_2(0) + \cdots + \zeta^n \varrho_n(0) + \cdots$$

with

$$\left.\begin{aligned} \varrho_0(0) &= f(0), \\ &\cdots\cdots, \\ \varrho_n(0) &= -\frac{1}{2\pi} \int\limits_{(S_1)} \varrho_{n-1}(1) \frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1 , \\ &\cdots\cdots\cdots\cdots \end{aligned}\right\} \tag{22}$$

From the properties of the iterated kernel we may write the nth equation in (22) in the form

$$\varrho_n(0) = \int_{(S_1)} K_n(1,0) f(1)\, d\sigma_1 \qquad (n = 1, 2, 3, \ldots). \tag{22'}$$

In the previous chapter we proved several theorems regarding ϱ_n which we shall use in what follows. In particular, we showed that ϱ_n has a limit in the ordinary case and in case (E). This limit is equal to zero if

$$\int_{(S)} f(0) d\sigma = 0 \qquad \text{in the ordinary case}$$

and if

$$\int_{(S^{(l)})} f(0) d\sigma = 0 \qquad (l = 1, 2, \ldots, k) \text{ in case } (E).$$

In case (J) ϱ_{2k} tends to a limit A, while ϱ_{2k-1} has a limit B which is in general different from A. $A + B$ is equal to zero if

$$\int_{(S)} f(0)\, d\sigma = 0;$$

$A - B$ is equal to zero if

$$\int_{(S^{(l)})} f(0)\, d\sigma = 0 \qquad (l = 1, 2, \ldots, k).$$

In all cases ϱ_n is bounded.

We now come to the proof of the theorem regarding the resolvent (15)

$$\frac{D(\zeta, 1, 0)}{D(\zeta)}$$

The resolvent $F(\zeta,1,0)$ of the equation

$$\varphi(0) = \zeta \int_{(S_1)} L(1,0) \varphi(1)\, d\sigma_1 + f(0)$$

satisfies the equality

$$F(\zeta, 1, 0) = \zeta \int_{(S_2)} L(2,0) F(\zeta, 1, 2)\, d\sigma_2 + L(1, 0);$$

from this it follows that the function (15) satisfies the equality

$$D(\zeta, 1, 0) = \zeta^n \int_{(S_2)} K_n(2,0) D(\zeta, 1, 2)\, d\sigma_2 + D(\zeta) K_n(1, 0)\,. \tag{23}$$

We suppose that the fraction (15) is in lowest form. Let ζ_0 be a pole of this fraction; then

$$D(\zeta_0, 1, 0) = \zeta_0^n \int_{(S_2)} K_n(2,0) D(\zeta_0, 1, 2)\, d\sigma_2\,. \tag{24}$$

Iterating one obtains:

$$
\begin{aligned}
D(\zeta_0, 1, 0) &= \zeta_0^n \int\limits_{(S_2)} K_n(2, 0) \left(\zeta_0^n \int\limits_{(S_3)} K_n(3, 2)\, D(\zeta_0, 1, 3)\, d\sigma_3 \right) d\sigma_2 \\
&= \zeta_0^{2n} \int\limits_{(S_3)} D(\zeta_0, 1, 3) \left(\int\limits_{(S_2)} K_n(3, 2)\, K_n(2, 0)\, d\sigma_2 \right) d\sigma_3 \\
&= \zeta_0^{2n} \int\limits_{(S_3)} K_{2n}(3, 0)\, D(\zeta_0, 1, 3)\, d\sigma_3 .
\end{aligned}
$$

Continuing these operations, we find:

$$D(\zeta_0, 1, 0) = \zeta_0^{ns} \int\limits_{(S_2)} K_{ns}(2, 0)\, D(\zeta_0, 1, 2)\, d\sigma_2 . \tag{24'}$$

If in (22) we substitute $\varrho_0 = f = D(\zeta_0, 1, 0)$,

then from (22′) follows

$$\varrho_{ns} = \int\limits_{(S_2)} K_{ns}(2, 0)\, D(\zeta_0, 1, 2)\, d\sigma_2 ,$$

which makes it possible to write (24′) in the form

$$D(\zeta_0, 1, 0) = \zeta_0^{ns} \varrho_{ns}(0) . \tag{25}$$

If we now assume that

$$|\zeta_0| < 1 ,$$

then letting s increase without bound in (25) we obtain:

$$D(\zeta_0, 1, 0) = 0; \tag{26}$$

this however contradicts the hypothesis made above that the fraction (15) was in lowest form.

If ζ_0 is not real with absolute value equal to one, then one can always find an odd number n satisfying the inequality $n\lambda > 2$ such that ζ_0^n is not equal to 1. In this case the right-hand side of (25) can tend to no other limit than zero as s increases without bound. Indeed, in the ordinary case, case (E), and case (J) $\varrho_{ns}(0)$ has a limit for odd s, while ζ_0^{ns} on the other hand does not. The limit of $\varrho_{ns}(0)$ must therefore be equal to zero. If this is the case, however, (26) then holds; but this leads to a contradiction.

Putting $\zeta_0 = -1$ in the ordinary case or in case (E), we obtain:

$$D(-1, 1, 0) = (-1)^s \varrho_{ns}(0) . \tag{25'}$$

In these cases $\varrho_{ns}(0)$ must tend to a limit as s increases without bound; hence, this limit must be equal to zero. We thus return to the equation

$$D(-1, 1, 0) = 0 ,$$

which contradicts the hypothesis made above.

Supposing that $\zeta_0 = 1$, and also $\zeta_0 = -1$ in case (J), does not lead to a contradiction.

We now wish to show that there can be no multiple poles at $\zeta = 1$ or $\zeta = -1$. Differentiating equation (23) with respect to ζ, we obtain:

$$D'(\zeta, 1, 0) = n\zeta^{n-1}\int\limits_{(S_2)} K_n(2, 0)\, D(\zeta, 1, 2)\, d\sigma_2 + \zeta^n \int\limits_{(S_2)} K_n(2, 0)\, D'(\zeta, 1, 2)\, d\sigma_2 + D'(\zeta)\, K_n(1, 0). \tag{27}$$

If we suppose that $\zeta = 1$ is a multiple pole, then $D'(1) = 0$, and from (27) it follows that

$$D'(1, 1, 0) = n\int\limits_{(S_2)} K_n(2, 0)\, D(1, 1, 2)\, d\sigma_2 + \int\limits_{(S_2)} K_n(2, 0)\, D'(1, 1, 2)\, d\sigma_2. \tag{28}$$

Let us treat first of all the ordinary case or the case (J). Integrating (28) over the entire boundary (S), we obtain:

$$\int\limits_{(S)} D'(1, 1, 0)\, d\sigma = n\int\limits_{(S)} \left(\int\limits_{(S_2)} K_n(2, 0)\, D(1, 1, 2)\, d\sigma_2 \right) d\sigma + \int\limits_{(S)} \left(\int\limits_{(S_2)} K_n(2, 0)\, D'(1, 1, 2)\, d\sigma_2 \right) d\sigma.$$

From the remarks in the preceding section it now follows that

$$\begin{aligned} \int\limits_{(S)} D'(1, 1, 0)\, d\sigma &= n\int\limits_{(S_2)} D(1, 1, 2) \left(\int\limits_{(S)} K_n(2, 0)\, d\sigma \right) d\sigma_2 \\ &\quad + \int\limits_{(S_2)} D'(1, 1, 2) \left(\int\limits_{(S)} K_n(2, 0)\, d\sigma \right) d\sigma_2 \\ &= n\int\limits_{(S_2)} D(1, 1, 2)\, d\sigma_2 + \int\limits_{(S_2)} D'(1, 1, 2)\, d\sigma_2. \end{aligned} \tag{29}$$

From the last equation it follows that

$$\int\limits_{(S)} D(1, 1, 0)\, d\sigma = 0. \tag{30}$$

If in (22) we now put

$$\varrho_0 = f = D(1,1,0),$$

we obtain from (30) for the ordinary case: $\lim\limits_{s\to\infty} \varrho_{ns} = 0$. From equation (25), which has the form

$$D(1,1,0) = \varrho_{ns}(0)$$

in the case under consideration, one finds that

$$D(1,1,0) = 0,$$

which leads to a contradiction.

In case (J)

$$A + B = 0$$

with

$$A = \lim_{t\to\infty} \varrho_{n\cdot 2t}, \qquad B = \lim_{t\to\infty} \varrho_{n(2t+1)};$$

from this it follows that

$$D(1,1,0) = A, \quad D(1,1,0) = B, \quad 2D(1,1,0) = 0, \quad D(1,1,0) = 0.$$

which is again impossible.

Now let us treat the case (E). We integrate equation (28) over the surfaces $(S^{(l)})$ $(l = 1, 2, \ldots, k)$. In place of equation (29) the following equations are then obtained:

$$\begin{aligned}\int_{(S^{(l)})} D'(1,1,0)\,d\sigma &= n\int_{(S_2)} D(1,1,2)\left(\int_{(S^{(l)})} K_n(2,0)\,d\sigma\right)d\sigma_2\\ &\quad+\int_{(S_2)} D'(1,1,2)\left(\int_{(S^{(l)})} K_n(2,0)\,d\sigma\right)d\sigma_2 \qquad (29')\\ &= n\int_{(S_2^{(l)})} D(1,1,2)\,d\sigma_2+\int_{(S_2^{(l)})} D'(1,1,2)\,d\sigma_2\\ &(l=1,2,\ldots,k).\end{aligned}$$

(This is because the integrals of the kernel K_n are different from zero only for points M_2 lying on $(S^{(l)})$.) Hence, the equations

$$\int_{(S^{(l)})} D(1,1,0)\,d\sigma = 0 \qquad (l=1,2,\ldots,k) \tag{30'}$$

hold. From these equations it follows that if

$$\varrho_0 = f = D(1,1,0),$$

then

$$\lim_{s\to\infty} \varrho_{ns} = 0, \qquad D(1,1,0) = 0,$$

i.e., we obtain a contradiction .Thus the hypothesis that $\zeta = 1$ is a multiple pole leads in all cases to a contradiction. Thus there is at most a simple pole at $\zeta = 1$.

We now consider the case (J) and assume that $\zeta = -1$ is a multiple pole. Since n is odd, we obtain from (27) in place of (28) the equation

$$\begin{aligned}D'(-1,1,0) &= n\int_{(S_2)} K_n(2,0)\,D(-1,1,2)\,d\sigma_2\\ &\quad-\int_{(S_2)} K_n(2,0)\,D'(-1,1,2)\,d\sigma_2 .\end{aligned}\tag{28'}$$

Integrating (29′) over the inner boundaries $(S^{(l)})$ $(l = 1, 2, \ldots, k)$ and taking into account the fact that the integral of the function $K_n(2,0)$ vanishes for all points M_2 not on $(S^{(l)})$ and has the value $(-1)^n = -1$ for points on $(S^{(l)})$, we find:

$$\begin{aligned}\int_{(S^{(l)})} D'(-1,1,0)\,d\sigma &= -n\int_{(S_2^{(l)})} D(-1,1,2)\,d\sigma_2+\int_{(S_2^{(l)})} D'(-1,1,2)\,d\sigma_2\\ &(l=1,2,\ldots,k).\end{aligned}\tag{29''}$$

From this it follows that

$$\int\limits_{(S_2^{(l)})} D(-1, 1, 2)\, d\sigma_2 = 0 \qquad (l = 1, 2, \ldots, k). \tag{30''}$$

If equations (30″) hold, then, putting $\varrho_0 = D(-1,1,0)$,

$$\lim_{t\to\infty} \varrho_{n\cdot 2t} = A, \qquad \lim_{t\to\infty} \varrho_{n(2t+1)} = B, \qquad A - B = 0.$$

But equation (25) now has the form

$$D(-1, 1, 0) = (-1)^s \varrho_{ns}.$$

Comparing this with the preceding expressions for the limits A and B, we obtain:

$$D(-1, 1, 0) = A, \quad D(-1, 1, 0) = -B,$$
$$2\, D(-1, 1, 0) = 0, \quad D(-1, 1, 0) = 0.$$

A contradiction is thus again obtained. Hence, $\zeta = -1$ is at most a simple pole.

§7. Two Lemmas Concerning an Integral Equation with Kernel $K_n(1,0)$

Lemma 1. *In the ordinary case or in case* (J) *the equation*

$$P(0) = \int\limits_{(S_1)} K_n(1, 0) P(1)\, d\sigma_1 + F(0) \tag{31}$$

has a solution if the equation

$$\int\limits_{(S)} F(0)\, d\sigma = 0 \tag{32}$$

holds.

Proof. The equation

$$P(0) = \zeta^n \int\limits_{(S_1)} K_n(1, 0)\, P(1)\, d\sigma_1 + F(0) \tag{33}$$

is solved by the function

$$P(0) = F(0) + \zeta^n \int\limits_{(S_1)} F(1) \frac{D(\zeta, 1, 0)}{D(\zeta)}\, d\sigma_1 = \frac{D_3(\zeta, 0)}{D_3(\zeta)}, \tag{34}$$

for function (15) is the resolvent of equation (33).

We shall assume that the fraction in the second part of (34) is in lowest form. To prove the lemma it then suffices to show that $\zeta = 1$ is not a pole of the fraction (34).

Integrating the identity

$$D_3(\zeta, 0) = \zeta^n \int\limits_{(S_1)} K_n(1, 0)\, D_3(\zeta, 1)\, d\sigma_1 + D_3(\zeta)\, F(0) \tag{33'}$$

over (S) and making use of condition (32), we find:

$$\begin{aligned}\int_{(S)} D_3(\zeta, 0)\, d\sigma &= \zeta^n \int_{(S)} \Big(\int_{(S_1)} K_n(1, 0)\, D_3(\zeta, 1)\, d\sigma_1 \Big) d\sigma \\ &= \zeta^n \int_{(S_1)} D_3(\zeta, 1) \Big(\int_{(S)} K_n(1, 0)\, d\sigma \Big) d\sigma_1 \\ &= \zeta^n \int_{(S_1)} D_3(\zeta, 1)\, d\sigma_1 .\end{aligned}$$

Then

$$(1 - \zeta^n) \int_{(S)} D_3(\zeta, 0)\, d\sigma = 0$$

and for $\zeta^n \neq 1$

$$\int_{(S)} D_3(\zeta, 0)\, d\sigma = 0 .$$

But since the last integral is an entire function of ζ, the last equation also holds for $\zeta = 1$. Hence,

$$\int_{(S)} D_3(1, 0)\, d\sigma = 0 . \tag{35}$$

We now put

$$\varrho_0 = f = D_3(1,0)$$

in (22). Then, just as in the preceding section, from the equation

$$D_3(1, 0) = \int_{(S_1)} K_n(1, 0)\, D_3(1, 1)\, d\sigma_1$$

which holds when $\zeta = 1$ is a pole of the fraction (34), we can deduce the validity of

$$D_3'(1,0) = \varrho_n = \varrho_{ns} ; \tag{36}$$

from condition (35) it follows that

$$D_3(1,0) = 0 .$$

Since, however, the fraction (34) is in lowest form, this equation is impossible. Hence, $\zeta = 1$ cannot be a pole of the fraction (34).

Lemma 1a. *In case* (E) *equation* (31) *has a solution if the* k *conditions*

$$\int_{(S^{(l)})} F(0)\, d\sigma = 0 \qquad (l = 1, 2, \ldots, k) \tag{32'}$$

are satisfied.

Proof. Integrating the indentity (33′) over $(S^{(l)})$, one obtains:

$$\int_{(S^{(l)})} D_3(\zeta, 0)\, d\sigma = \zeta^n \int_{(S_1{}^{(l)})} D_3(\zeta, 1)\, d\sigma_1 \qquad (l = 1, 2, \ldots, k),$$

since the integral of the function $K_n(1,0)$ which here enters the computation is nonzero only for points M_1 lying on $(S^{(l)})$. From this we can conclude that

$$\int_{(S^{(l)})} D_3(\zeta, 0)\, d\sigma = 0 \qquad \text{for} \qquad \zeta^n \neq 1 \qquad (l = 1, 2, \ldots, k).$$

Then also

$$\int_{(S^{(l)})} D_3(1, 0)\, d\sigma = 0 \qquad (l = 1, 2, \ldots, k).$$

If now the k conditions are satisfied and if we assume that $\zeta = 1$ is a pole of (34), then it follows, on putting $\varrho_0 = D_3(1,0)$, that

$$D_3(1, 0) = \lim_{s \to \infty} \varrho_{n_s} = 0\,,$$

which is impossible.

Lemma 2. *The equation*

$$P(0) = -\int_{(S_1)} K_n(1, 0)\, P(1)\, d\sigma_1 + F(0) \tag{31'}$$

has a solution in case (J) *if the equations*

$$\int_{(S^{(l)})} F(0)\, d\sigma = 0 \quad (l = 1, 2, \ldots, k). \tag{32'}$$

hold.

Proof. We shall show that if conditions (32′) hold, then there can be no pole at $\zeta = -1$.

Integrating the identity (33′) over $(S^{(l)})$, we obtain:

$$\int_{(S^{(l)})} D_3(\zeta, 0)\, d\sigma = \zeta^n \int_{(S_1)} D_3(\zeta, 1) \left(\int_{(S^{(l)})} K_n(1, 0)\, d\sigma \right) d\sigma_1$$
$$= -\zeta^n \int_{(S_1^{(l)})} D_3(\zeta, 1)\, d\sigma_1;$$

the integral in parentheses vanishes if M_1 does not lie on $(S^{(l)})$ and for points M_1 on $(S^{(l)})$ equals

$$(-1)^n = -1\,.$$

From the equation

$$(1 + \zeta^n) \int_{(S^{(l)})} D_3(\zeta, 0)\, d\sigma = 0$$

one concludes that

$$\int_{(S^{(l)})} D_3(\zeta, 0)\, d\sigma = 0 \qquad \text{for} \qquad \zeta^n \neq -1 \qquad (l = 1, 2, \ldots, k).$$

Since this integral is an entire function, it follows that

$$\int_{(S^{(l)})} D_3(-1, 0)\, d\sigma = 0 \quad (l = 1, 2, \ldots, k). \tag{35'}$$

If $\zeta = -1$ is a pole of the fraction (34), then

$$D_3(-1, 0) = (-1)^n \int_{(S_1)} K_n(1, 0)\, D_3(-1, 1)\, d\sigma_1 .$$

Putting

$$\varrho_0 = f = D_3(-1, 0)$$

in (22), one obtains:

$$D_3(-1, 0) = (-1)^{ns} \varrho_{ns} .$$

Hence,

$$D_3(-1, 0) = \lim_{t \to \infty} \varrho_{n \cdot 2t} = A ,$$

$$D_3(-1, 0) = \lim_{t \to \infty} (-\varrho_{n(2t+1)}) = -B ;$$

from condition (35′) $A - B = 0$, so that the equality

$$D_3(-1, 0) = 0$$

results; but this is a contradiction.

Since $\zeta = -1$ is also not a pole of the fraction (34), we can determine $P(0)$ by putting $\zeta = -1$ in formula (34).

§8. Two Lemmas on the Potential of the Double Layer

Lemma 1. *If the function* $\overline{W}(0)$ *defined on* (S) *satisfies the equation*

$$\overline{W}(0) = \frac{1}{2\pi} \int_{(S_1)} \frac{\cos(r_{10} N_1)}{r_{10}^2} \overline{W}(1)\, d\sigma_1 , \tag{37}$$

then $\overline{W}(0)$ *is constant on* (S); *in the ordinary case and in case* (J) $\overline{W}(0)$ *has the same value on the entire boundary, and in case* (E) $\overline{W}(0)$ *has a particular value* $C^{(l)}$ *on each boundary surface* $(S^{(l)})$.

Proof. Writing equation (37) in the form

$$\overline{W}(0) = \int_{(S_1)} K(0, 1) \overline{W}(1)\, d\sigma_1$$

and applying the iteration procedure, we obtain:

$$\begin{aligned} \overline{W}(0) &= \int_{(S_1)} K(0, 1) \overline{W}(1)\, d\sigma_1 = \int_{(S_1)} K_2(0, 1) \overline{W}(1)\, d\sigma_1 \\ &= \cdots = \int_{(S_1)} K_n(0, 1) \overline{W}(1)\, d\sigma_1 . \end{aligned} \tag{37′}$$

We first consider the case (J) and the ordinary case. We take two arbitrary points M_2 and M_3 on (S); here M_3 need not necessarily lie on the same boundary surface as M_2 (Fig. 27).

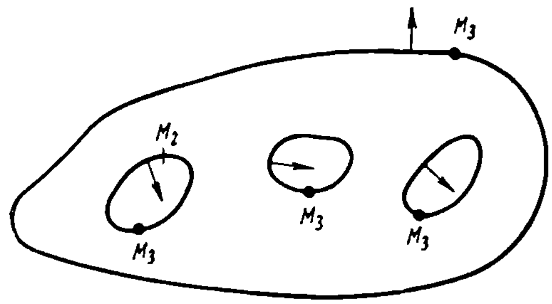

Fig. 27

We put

$$F(0) = K_n(3, 0) - K_n(2, 0).$$

From the remarks in §5

$$\int\limits_{(S)} F(0)\, d\sigma = \int\limits_{(S)} K_n(3, 0)\, d\sigma - \int\limits_{(S)} K_n(2, 0)\, d\sigma = 1 - 1 = 0;$$

therefore, the equation

$$P(0) = \int\limits_{(S_1)} K_n(1, 0)\, P(1)\, d\sigma_1 + K_n(3, 0) - K_n(2, 0) \tag{38}$$

has a solution in the ordinary case and in case (J) on the basis of Lemma 1 of the preceding section.

Writing (37′) in the form

$$\overline{W}(1) = \int\limits_{(S)} K_n(1, 0)\, \overline{W}(0)\, d\sigma,$$

multiplying $\overline{W}(1)$ by $P(1)$, and integrating the product over (S), we obtain on making use of (38):

$$\begin{aligned}
\int\limits_{(S_1)} \overline{W}(1)\, P(1)\, d\sigma_1 &= \int\limits_{(S_1)} P(1) \left(\int\limits_{(S)} K_n(1, 0)\, \overline{W}(0)\, d\sigma \right) d\sigma_1 \\
&= \int\limits_{(S)} \overline{W}(0) \left(\int\limits_{(S_1)} K_n(1, 0)\, P(1)\, d\sigma_1 \right) d\sigma \\
&= \int\limits_{(S)} \overline{W}(0) \{P(0) - K_n(3, 0) + K_n(2, 0)\}\, d\sigma \\
&= \int\limits_{(S)} \overline{W}(0)\, P(0)\, d\sigma - \int\limits_{(S)} K_n(3, 0)\, \overline{W}(0)\, d\sigma \\
&\quad + \int\limits_{(S)} K_n(2, 0)\, \overline{W}(0)\, d\sigma.
\end{aligned}$$

From this it follows that

$$\int\limits_{(S)} K_n(3, 0)\, \overline{W}(0)\, d\sigma = \int\limits_{(S)} K_n(2, 0)\, \overline{W}(0)\, d\sigma$$

and further

$$\overline{W}(3) = \overline{W}(2). \tag{39}$$

This means that the value of $\overline{W}(0)$ at an arbitrary point M_3 is equal to its value at M_2, i.e.,

$$\overline{W}(0) = C.$$

We recall that because of the properties of GAUSS' integral a constant is indeed a solution of equation (37).

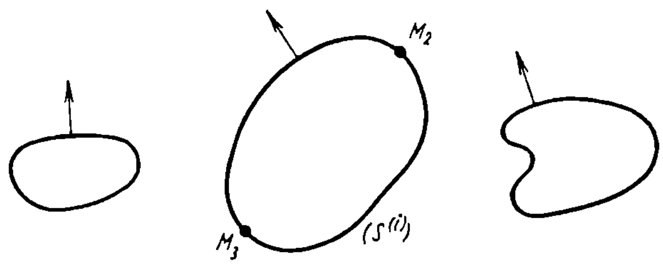

Fig. 28

We now investigate case (E). For this we consider two points M_2 and M_3 on the same surface $(S^{(i)})$ (Fig. 28) and form the function

$$F(0) = K_n(3,0) - K_n(2,0).$$

For each of the boundary surfaces $(S^{(l)})$

$$\int\limits_{(S^{(l)})} F(0)\, d\sigma = \int\limits_{(S^{(l)})} K_n(3, 0)\, d\sigma - \int\limits_{(S^{(l)})} K_n(2, 0)\, d\sigma = 0,$$

since both integrals vanish for $l \neq i$ and are both equal to one for $l = i$.

Equation (38) therefore has a solution, and we see that condition (39) is also satisfied here; hence, on $(S^{(i)})$

$$\overline{W}(0) = C^{(i)}.$$

If the equality

$$\overline{W}(0) = C^{(l)} \quad \text{on } (S^{(l)}) \qquad (l = 1, 2, \ldots, k)$$

holds, then from the properties of GAUSS' integral for a point M_1 on $(S^{(i)})$

$$\frac{1}{2\pi} \int\limits_{(S_1)} \frac{\cos(r_{10} N_1)}{r_{10}^2} \overline{W}(1)\, d\sigma_1 = \sum_{l=1}^{k} \frac{C^{(l)}}{2\pi} \int\limits_{(S_1^{(l)})} \frac{\cos(r_{10} N_1)}{r_{10}}\, d\sigma_1 = C^{(i)},$$

since in the last sum only the integral over $(S^{(i)})$ is different from zero, and this has the value 2π. We have thus found the solution of equation (37) for the case (E).

Lemma 2. *If in case (J) the function* $\overline{W}(0)$ *satisfies the equation*

$$\overline{W}(0) = -\frac{1}{2\pi}\int_{(S_1)} \frac{\cos(r_{10} N_1)}{r_{10}^2} \overline{W}(1)\, d\sigma_1, \tag{40}$$

then it is constant on each boundary surface. On the outer boundary surface $(S^{(0)})$ *it is equal to zero; the value of the function may change when passing from one boundary surface to another.*

Proof. Writing equation (40) in the form

$$\overline{W}(0) = -\int_{(S_1)} K(0,1)\overline{W}(1)\, d\sigma_1$$

and applying the iteration procedure, we obtain:

$$\begin{aligned}\overline{W}(0) &= -\int_{(S_1)} K(0,1)\,\overline{W}(1)\, d\sigma_1 = (-1)^2\int_{(S_1)} K_2(0,1)\overline{W}(1)\, d\sigma_1 \\ &= \cdots = (-1)^n \int_{(S_1)} K_n(0,1)\,\overline{W}(1)\, d\sigma_1 .\end{aligned} \tag{40'}$$

Let M_2 and M_3 be two points of an inner boundary surface $(S^{(i)})$ (Fig. 29);

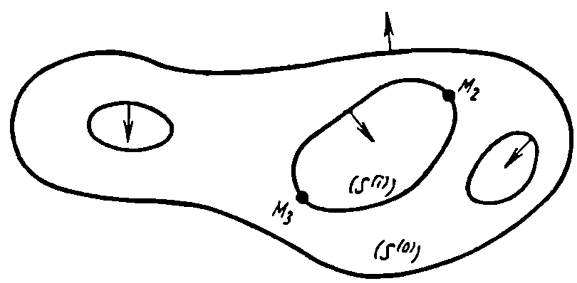

Fig. 29

we consider the function

$$F(0) = K_n(3,0) - K_n(2,0).$$

For every inner boundary surface $(S^{(l)})$

$$\int_{(S^{(l)})} F(0)\, d\sigma = \int_{(S^{(l)})} K_n(3,0)\, d\sigma - \int_{(S^{(l)})} K_n(2,0)\, d\sigma = 0$$

$$(l = 1, 2, \ldots, k),$$

since these integrals both vanish for $l \neq i$, while for $l = i$ they are equal to $(-1)^n$. From this it follows that the equation

$$P(0) = -\int_{(S_1)} K_n(1,0)\, P(1)\, d\sigma_1 + K_n(3,0) - K_n(2,0) \tag{41}$$

has a solution if n is odd. Writing (40') in the form

$$\overline{W}(1) = -\int_{(S)} K_n(1,0)\,\overline{W}(0)\, d\sigma,$$

multiplying this equation by $P(1)$, and integrating the product obtained over the entire boundary (S), we find with the help of (41) that

$$\begin{aligned}\int\limits_{(S_1)} \overline{W}(1)\,P(1)\,d\sigma_1 &= -\int\limits_{(S)} \overline{W}(0)\left(\int\limits_{(S_1)} K_n(1,0)\,P(1)\,d\sigma_1\right)d\sigma \\ &= \int\limits_{(S)} \overline{W}(0)\,P(0)\,d\sigma - \int\limits_{(S)} K_n(3,0)\,\overline{W}(0)\,d\sigma \\ &\quad + \int\limits_{(S)} K_n(2,0)\,\overline{W}(0)\,d\sigma\,;\end{aligned}$$

hence,

$$\int\limits_{(S)} K_n(3,0)\,\overline{W}(0)\,d\sigma = \int\limits_{(S)} K_n(2,0)\,\overline{W}(0)\,d\sigma,$$

and further

$$\overline{W}(3) = \overline{W}(2).$$

Therefore, $\overline{W}(0)$ has the constant value

$$\overline{W}(0) = C^{(i)}$$

on the boundary surface $(S^{(i)})$. From this it follows for a point M_0 on $(S^{(0)})$ that

$$\begin{aligned}\overline{W}(0) &= -\frac{1}{2\pi}\int\limits_{(S_1^{(0)})} \frac{\cos(r_{10}N_1)}{r_{10}^2}\,\overline{W}(1)\,d\sigma_1 + \sum_{l=1}^{k} C^{(l)} \int\limits_{(S_1^{(l)})} \frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1 \\ &= -\frac{1}{2\pi}\int\limits_{(S_1^{(0)})} \frac{\cos(r_{10}N_1)}{r_{10}^2}\,\overline{W}(1)\,d\sigma_1\,;\end{aligned}$$

but the equation

$$\overline{W}(0) = -\frac{1}{2\pi}\int\limits_{(S_1^{(0)})} \frac{\cos(r_{10}N_1)}{r_{10}^2}\,\overline{W}(1)\,d\sigma_1$$

for the ordinary case has only the solution

$$\overline{W}(0) = 0$$

on $(S^{(0)})$, since in the ordinary case $\zeta = -1$ is not an eigenvalue.

Conversely, if $\overline{W}(0)$ is equal to $C^{(l)}$ on the boundary surface $(S^{(l)})$ ($l = 1, 2, \ldots, k$) and equal to zero on $(S^{(0)})$, then it is easily shown that $\overline{W}(0)$ is a solution of equation (40). Indeed, for a point M_0 on $(S^{(i)})$

$$\frac{1}{2\pi}\int\limits_{(S_1)} \frac{\cos(r_{10}N_1)}{r_{10}^2}\,\overline{W}(1)\,d\sigma_1 = \sum_{l=1}^{k} \frac{C^{(l)}}{2\pi}\int\limits_{(S_1^{(l)})} \frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1 = -C^{(i)},$$

since for this point only the integral over $(S^{(i)})$ is different from zero; its value is -2π, since the normals of $(S^{(i)})$ are directed into the interior of $(S^{(i)})$.

§9. Consequences of the Lemmas of §8

From the lemmas of §8 we have found all the eigenfunctions of equation (5) corresponding to the eigenvalues $\zeta = -1$ and $\zeta = 1$.

In the ordinary case and in case (J) there exists only one linearly independent eigenfunction corresponding to the eigenvalue $\zeta = 1$; the most general eigenfunction is equal to C. In case (E) there are k linearly independent eigenfunctions corresponding to the eigenvalue $\zeta = 1$; one may assume that they satisfy the following conditions:

$$\varphi^{(l)} = \alpha^{(l)} \neq 0 \ \text{ on } \ (S^{(l)}), \quad \varphi^{(l)} = 0 \ \text{ on } \ (S^{(i)})$$
$$(i \neq l, \ l = 1, 2, \ldots, k).$$

In case (J) there are k eigenfunctions corresponding to the eigenvalue $\xi = 1$ which can be selected such that the following conditions are satisfied:

$$\varphi^{(l)} = \alpha^{(l)} \neq 0 \ \text{ on } \ (S^{(l)}), \quad \varphi^{(l)} = 0 \ \text{ on } \ (S^{(i)})$$
$$(i \neq l, \ l = 1, 2, \ldots, k).$$

This now clarifies the results obtained in the previous chapter. We know that the integral equation (12)

$$\mu(0) = -\frac{\zeta}{2\pi} \int\limits_{(S_1)} \frac{\cos(r_{10} N_0)}{r_{10}^2} \mu(1)\, d\sigma_1 = f(0),$$

with ζ an eigenvalue has a solution only if the equation

$$\int\limits_{(S)} f(0)\, \varphi(0)\, d\sigma = 0$$

is satisfied, where $\varphi(0)$ is the general solution of the homogeneous equation associated to (12) corresponding to the eigenvalue ζ.

From this it follows that in the ordinary case and in case (J) for $\zeta = 1$ the condition

$$\int\limits_{(S)} f(0)\, d\sigma = 0,$$

and in case (E) the conditions

$$\int\limits_{(S)} f(0)\, \varphi(0)\, d\sigma = \int\limits_{(S^{(l)})} f(0)\, C^{(l)} d\sigma = 0 \qquad (C^{(l)} \neq 0),$$

i.e.,

$$\int\limits_{(S^{(l)})} f(0)\, d\sigma = 0 \qquad (l = 1, 2, \ldots, k)$$

must be satisfied; finally, for $\zeta = -1$ in case (J) conditions of the same form as the conditions for $\zeta = 1$ in case (E) must hold.

§10. The Solution of the Inner Dirichlet Problem for Case (E) and the Ordinary Case

The considerations in §6 show that we are able to solve the inner Dirichlet problem in case (E) and in the ordinary case. The number $\zeta = -1$ is not a pole of the function $\vartheta(0)$ and hence also not a pole of the series (8)

$$W = W_1 + \zeta W_2 + \zeta^2 W_3 + \cdots$$

with

$$W_1 = \frac{1}{2\pi}\int_{(S_1)} f \frac{\cos(r_{10} N_1)}{r_{10}^2}\, d\sigma_1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$W_n = \frac{1}{2\pi}\int_{(S_1)} \overline{W}_{n-1} \frac{\cos(r_{10} N_1)}{r_{10}^2}\, d\sigma_1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

It is hereby assumed that there is no pole of the function $\vartheta(0)$ at $\zeta = 1$.

The radius of convergence of the series (8) is equal to one if $\zeta = 1$ is a pole of the function $\vartheta(0)$. Since this pole must be simple, the radius of convergence of the series

$$(1-\zeta) W = W_1 + \zeta(W_2 - W_1) + \zeta^2(W_3 - W_2) + \cdots + \zeta^n(W_{n+1} - W_n) + \cdots$$

is always greater than one. If we replace ζ by -1 we obtain the function

$$W = \tfrac{1}{2}[W_1 - (W_2 - W_1) + (W_3 - W_2) - + \cdots],$$

which from remarks in §2 satisfies the condition

$$W_i = f \quad \text{on} \quad (S).$$

§11. Investigation of the Pole $\zeta = 1$ in Case (E) and in the Ordinary Case

We shall here treat exclusively the case (E), since the ordinary case is obtained as a special case for $k = 1$. As we have found in §4, the function (17)

$$\vartheta(0) = \frac{D_2(\zeta, 0)}{D_2(\zeta)}$$

is a solution of equation (5)

$$\vartheta(0) = \frac{\zeta}{2\pi}\int_{(S_1)} \vartheta(1) \frac{\cos(r_{10} N_1)}{r_{10}^2}\, d\sigma_1 + f(0).$$

We shall now determine under what conditions $\zeta = 1$ is not a pole of this function.

The number $\zeta = 1$ is an eigenvalue of equation (12)

$$\mu(0) = -\frac{\zeta}{2\pi}\int_{(S_1)} \mu(1)\frac{\cos(r_{10} N_0)}{r_{10}^2} d\sigma_1 + f(0)$$

associated to equation (5). Hence, equation (5) has only one solution for $\zeta = 1$ if the equation

$$\int_{(S)} f(0)\,\varphi(0)\,d\sigma = 0$$

is satisfied; here $\varphi(0)$ is the general solution of the homogeneous equation (12) corresponding to the eigenvalue $\zeta = 1$.

In III, §4 we found all the eigenfunctions of equation (12) belonging to the eigenvalue $\zeta = 1$; among these there are k linearly independent functions $\varrho^{(\lambda)}$ ($\lambda = 1, 2, \ldots, k$) characterized by the following conditions:

$$\int_{(S^{(l)})} \varrho^{(\lambda)} d\sigma = 0 \qquad (l \neq \lambda); \qquad \int_{(S^{(\lambda)})} \varrho^{(\lambda)} d\sigma = 1;$$

$$\varrho^{(\lambda)}(0) = -\frac{1}{2\pi}\int_{(S_1)} \varrho^{(\lambda)}(1)\,\frac{\cos(r_{10} N_0)}{r_{10}^2}\, d\sigma_1.$$

In the ordinary case there is only one eigenfunction which we denoted by ϱ.

The k equations

$$\int_{(S)} f\varrho^{(l)} d\sigma = 0 \qquad (l = 1, 2, \ldots, k) \tag{42}$$

thus constitute the necessary and sufficient conditions for the existence of a solution of equation (5) for $\zeta = 1$.

Without making use of the results from the theory of integral equations just cited, one can immediately show that if conditions (42) hold then the function (17) has no pole at $\zeta = 1$.

Putting (17) into equation (5), one finds:

$$D_2(\zeta, 0) = \frac{\zeta}{2\pi}\int_{(S_1)} D_2(\zeta, 1)\,\frac{\cos(r_{10} N_1)}{r_{10}^2}\, d\sigma_1 + D_2(\zeta) f(0). \tag{43}$$

We shall assume that conditions (42) are satisfied. Multiplying both sides of (43) by $\varrho^{(\lambda)}$ and integrating the product so obtained over the entire boundary (S), we find:

$$\int\limits_{(S)} D_2(\zeta, 0)\, \varrho^{(\lambda)}(0)\, d\sigma$$

$$= \frac{\zeta}{2\pi} \int\limits_{(S)} \varrho^{(\lambda)}(0) \left(\int\limits_{(S_1)} D_2(\zeta, 1) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 \right) d\sigma + D_2(\zeta) \int\limits_{(S)} f(0)\, \varrho^{(\lambda)} d\sigma$$

$$= \frac{\zeta}{2\pi} \int\limits_{(S_1)} D_2(\zeta, 1) \left(\int\limits_{(S)} \frac{\cos(r_{10} N_1)}{r_{10}^2} \varrho^{(\lambda)} d\sigma \right) d\sigma_1$$

$$= -\frac{\zeta}{2\pi} \int\limits_{(S_1)} D_2(\zeta, 1) \left(\int\limits_{(S)} \frac{\cos(r_{01} N_1)}{r_{10}^2} \varrho^{(\lambda)} d\sigma \right) d\sigma_1$$

$$= \zeta \int\limits_{(S_1)} D_2(\zeta, 1)\, \varrho^{(\lambda)}(1)\, d\sigma_1;$$

whence it follows that

$$(1 - \zeta) \int\limits_{(S)} D_2(\zeta, 0)\, \varrho^{(\lambda)} d\sigma = 0.$$

Thus for $\zeta \neq 1$

$$\int\limits_{(S)} D_2(\zeta, 0)\, \varrho^{(\lambda)} d\sigma = 0.$$

Since the integral on the left-hand side of the last equation is an entire function of ζ, this equation also holds for $\zeta = 1$, i.e.,

$$\int\limits_{(S)} D_2(1, 0)\, \varrho^{(\lambda)}(0)\, d\sigma = 0 \qquad (\lambda = 1, 2, \ldots, k). \tag{44}$$

Now let $\zeta = 1$ be a pole of the function (17); replacing ζ in (43) by 1, we obtain:

$$D_2(1, 0) = \frac{1}{2\pi} \int\limits_{(S_1)} D_2(1, 1) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1. \tag{45}$$

With the help of Lemma 1 of §8 we deduce from equation (45) that on the surface $(S^{(l)})$

$$D_2(1,0) = C^{(l)} \qquad (l = 1, 2, \ldots, k).$$

Substituting this value for $D_2(1,0)$ into (44), we obtain:

$$\int\limits_{(S)} D_2(1, 0)\, \varrho^{(\lambda)}(0)\, d\sigma = \sum_{l=1}^{k} \int\limits_{(S^{(l)})} D_2(1, 0)\, \varrho^{(\lambda)} d\sigma$$

$$= \sum_{l=1}^{k} C^{(l)} \int\limits_{(S^{(l)})} \varrho^{(\lambda)}(0)\, d\sigma = C^{(\lambda)} = 0$$

$$(\lambda = 1, 2, \ldots, k).$$

From this it follows that $D_2(1,0)$ is equal to zero and that contrary to hypothesis the fraction (17) is not in lowest form. We thus arrive at the conclusion that $\zeta = 1$ is not a pole of the function (17).

If conditions (42) hold we can now obtain the solution of the outer DIRICHLET problem. In this case the function W does not have a pole at $\zeta = 1$, and the radius of convergence of the series (8) is greater than one; the function

$$W = W_1 + W_2 + \cdots \tag{46}$$

is the solution of the DIRICHLET problem with the property that

$$W_e = -f.$$

Conditions (42) do not however stand in any direct relation to Problem A.

§12. Interpretation of Conditions (42)

We solved the outer DIRICHLET problem after imposing conditions on the values of the function on the boundary which are entirely foreign to the problem. In this section we shall show that conditions (42) are necessary for the solvability of Problem (B_e).

Theorem. *For each double-layer potential*

$$W = \int\limits_{(S_1)} \mu(1) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1$$

and for each eigenfunction

$$\int\limits_{(S)} W_e \varrho^{(\lambda)} d\sigma = 0.$$

Proof. Supposing that the proof has been completed, one sees that it is impossible to solve Problem (B_e) if the conditions (42) are not satisfied.

To prove the theorem, we note first of all that

$$W_e = \left(\int\limits_{(S_1)} \mu(1) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 \right)_e = \int\limits_{(S_1)} \mu(1) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 - 2\pi\mu(0);$$

from this it follows immediately that

$$\begin{aligned}\int\limits_{(S)} W_e \varrho^{(\lambda)} d\sigma &= \int\limits_{(S)} \varrho^{(\lambda)} \Bigg(\int\limits_{(S_1)} \mu(1) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 \Bigg) d\sigma - 2\pi \int\limits_{(S)} \mu(0) \varrho^{(\lambda)} d\sigma \\ &= \int\limits_{(S_1)} \mu(1) \Bigg(\int\limits_{(S)} \varrho^{(\lambda)} \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma \Bigg) d\sigma_1 - 2\pi \int\limits_{(S)} \mu(0) \varrho^{(\lambda)} d\sigma \\ &= \int\limits_{(S_1)} \mu(1) \Bigg(- \int\limits_{(S)} \varrho^{(\lambda)} \frac{\cos(r_{01} N_1)}{r_{10}^2} d\sigma \Bigg) d\sigma_1 - 2\pi \int\limits_{(S)} \mu(0) \varrho^{(\lambda)} d\sigma \\ &= 2\pi \int\limits_{(S_1)} \mu(1) \varrho^{(\lambda)}(1) d\sigma_1 - 2\pi \int\limits_{(S)} \mu(0) \varrho^{(\lambda)}(0) d\sigma = 0,\end{aligned}$$

as was to be shown.

§13. The Solution of the Outer Problem for Case (E)

We now make use of the results found in §11; let α be a function of the points of (S) which is constant on each surface $(S^{(l)})$:

$$\alpha = \alpha^{(l)} \quad \text{on} \quad (S^{(l)});$$

let the constants $\alpha^{(l)}$ be chosen in such a manner that the function $f - \alpha$ satisfies conditions (42).

Since

$$\int\limits_{(S)} (f - \alpha) \varrho^{(\lambda)} d\sigma = \int\limits_{(S)} f \varrho^{(\lambda)} d\sigma - \sum_{l=1}^{k} \alpha^{(l)} \int\limits_{(S^{(l)})} \varrho^{(\lambda)} d\sigma = \int\limits_{(S)} f \varrho^{(\lambda)} d\sigma - \alpha^{(\lambda)},$$

we must put

$$\alpha^{(l)} = \int\limits_{(S)} f \varrho^{(l)} d\sigma \qquad (l = 1, 2, \ldots, k). \tag{47}$$

Let w be the solution of Problem (B_e) corresponding to the function $f - \alpha$; from §11 one has:

$$w = w_1 + w_2 + w_3 + \cdots, \quad w_e = \alpha - f \tag{48}$$

with

$$\left.\begin{aligned} w_1 &= \frac{1}{2\pi} \int\limits_{(S_1)} (f - \alpha) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots, \\ w_k &= \frac{1}{2\pi} \int\limits_{(S_1)} \overline{w}_{k-1} \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1, \\ &\cdots\cdots\cdots\cdots\cdots\cdots. \end{aligned}\right| \tag{49}$$

For points in the interior of (D_e) the first equation of (49) gives us

$$w_1 = \frac{1}{2\pi} \iint\limits_{(S_1)} \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 - \sum_{l=1}^{k} \frac{\alpha^{(l)}}{2\pi} \int\limits_{(S_1^{(l)})} \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 = W_1 ,$$

while for points of (S)

$$\overline{w}_1 = \frac{1}{2\pi} \iint\limits_{(S_1)} \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 - \sum_{l=1}^{k} \frac{\alpha^{(l)}}{2\pi} \int\limits_{(S_1^{(l)})} \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 = \overline{W}_1 - \alpha ;$$

the integral

$$\int\limits_{(S_1^{(l)})} \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1$$

is equal to zero for M_0 in the interior of (D_e) or on one of the surfaces $(S^{(\lambda)})$ with $\lambda \neq l$ and equals 2π if M_0 belongs to the surface $(S^{(l)})$. Making use of the value found for w_1 to compute w_2 and proceeding in this manner, we obtain successively:

$$w_2 = W_2 , \quad \overline{w}_2 = \overline{W}_2 - \alpha , \quad w_3 = W_3 , \quad \overline{w}_3 = \overline{W}_3 - \alpha , \quad \ldots .$$

From this it follows that the series (46) and (48) do not differ inside (D_e) and that the series (48) converges in this region only if conditions (42) are not satisfied; in this case, however,

$$W_e = \alpha - f$$

(α is defined by equations (47)).

In III, §9 we proved that each potential

$$\int\limits_{(S_1)} \frac{\varrho^{(l)} d\sigma_1}{r_{10}} \tag{50}$$

is constant inside each surface $(S^{(\lambda)})$. Let $C_\lambda^{(l)}$ be the value of the potential (50) in the interior of $(S^{(\lambda)})$. One can always choose numbers

$$\gamma_1, \gamma_2, \ldots, \gamma_k$$

such that the following k equations are satisfied:

$$\sum_{l=1}^{k} \gamma_l \int\limits_{(S_1)} \frac{\varrho^{(l)} d\sigma_1}{r_{10}} = \alpha^{(\lambda)} \quad \text{on} \quad (S^{(\lambda)}) \qquad (\lambda = 1, 2, \ldots, k) . \tag{51}$$

These equations are equivalent to

$$\sum_{l=1}^{k} \gamma_l C_\lambda^{(l)} = \alpha^{(\lambda)} . \tag{51'}$$

If the system (51′) had no solution for a certain set of constants $\alpha^{(1)}, \ldots, \alpha^{(k)}$ then one could find constants $\delta_1, \ldots, \delta_k$ not all zero satisfying the conditions

$$\sum_{l=1}^{k} \delta_l C_\lambda^{(l)} = 0 \qquad (\lambda = 1, 2, \ldots, k).$$

For such constants $\delta_1, \ldots, \delta_k$ the potential

$$\int_{(S_1)} \sum_{l=1}^{k} \delta_l \varrho^{(l)} \frac{d\sigma_1}{r_{10}}$$

would be zero in each of the regions bounded by the surfaces $(S^{(l)})$; it would therefore be everywhere zero and would imply that

$$\sum_{l=1}^{k} \delta_l \varrho^{(l)} = 0,$$

which contradicts the linear independence of the functions $\varrho^{(l)}$. From this it follows that one can indeed find numbers $\gamma_1, \gamma_2, \ldots, \gamma_k$ with the required property. For points on (S) equations (51) demonstrate the validity of the equation

$$V_0 = \int_{(S_1)} \sum_{l=1}^{k} \gamma_l \varrho^{(l)} \frac{d\sigma_1}{r_{10}} = \alpha.$$

We put

$$V = V_0 - W. \tag{52}$$

Computing V_e, one finds:

$$V_e = (V_0)_e - W_e = \alpha - \alpha + f = f;$$

from this it follows that the function V is the solution of the problem in question. In the case at hand the solution of Problem A is given as the sum of a simple-layer potential and a double-layer potential.

§14. The Case (J). The Conditions that $\zeta = -1$ Not Be a Pole

It remains to consider the Dirichlet problem for case (J). We turn first to the inner problem. To solve the corresponding Problem B, one must find the solution of equation (5)

$$\vartheta(0) = \frac{\zeta}{2\pi} \int_{(S_1)} \vartheta(1) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 + f(0)$$

for $\zeta = -1$. In the case at hand this solution is given by the value of the function (17)

$$\vartheta(0) = \frac{D_2(\zeta, 0)}{D_2(\zeta)}$$

for $\zeta = -1$ if there is no pole here.

We proved in III, §17 that corresponding to the eigenvalue $\zeta = -1$ the equation (12)

$$\mu(0) = -\frac{\zeta}{2\pi}\int_{(S_1)} \mu(1)\frac{\cos(r_{10}\,N_0)}{r_{10}^2}\,d\sigma_1 + f(0)$$

has k linearly independent eigenfunctions $\psi^{(\lambda)}$ characterized by the following conditions:

$$\int_{(S^{(l)})} \psi^{(\lambda)}(0)\,d\sigma = 0\,, \qquad \text{if} \quad l \neq \lambda\,,$$

$$\int_{(S^{(\lambda)})} \psi^{(\lambda)}(0)\,d\sigma = 1 \qquad\qquad (\lambda = 1, 2, \ldots, k)$$

and

$$\psi^{(\lambda)}(0) = \frac{1}{2\pi}\int_{(S_1)} \psi^{(\lambda)}(1)\frac{\cos(r_{10}\,N_0)}{r_{10}^2}\,d\sigma_1\,.$$

From this it follows that equation (5) has a solution for $\zeta = -1$ only if the following conditions are satisfied:

$$\int_{(S)} f(0)\,\psi^{(\lambda)}(0)\,d\sigma = 0 \qquad (\lambda = 1, 2, \ldots, k)\,. \tag{53}$$

If these conditions are satisfied, then $\zeta = -1$ cannot be a pole of the function $\vartheta(0)$. Multiplying both sides of the equation

$$D_2(\zeta, 0) = \frac{\zeta}{2\pi}\int_{(S_1)} D_2(\zeta, 1)\frac{\cos(r_{10}\,N_1)}{r_{10}^2}\,d\sigma_1 + D_2(\zeta)\,f(0)$$

by $\psi^{(\lambda)}(0)$ and integrating the product over the entire boundary (S), we obtain on making use of (53):

$$\begin{aligned}\int_{(S)} D_2(\zeta, 0)\,\psi^{(\lambda)}(0)\,d\sigma &= \frac{\zeta}{2\pi}\int_{(S)} \psi^{(\lambda)}(0)\left(\int_{(S_1)} D_2(\zeta, 1)\frac{\cos(r_{10}\,N_1)}{r_{10}^2}\,d\sigma_1\right)d\sigma \\ &= -\frac{\zeta}{2\pi}\int_{(S_1)} D_2(\zeta, 1)\left(\int_{(S)} \psi^{(\lambda)}(0)\frac{\cos(r_{01}\,N_1)}{r_{10}^2}\,d\sigma\right)d\sigma_1 \\ &= -\zeta\int_{(S_1)} D_2(\zeta, 1)\,\psi^{(\lambda)}(1)\,d\sigma\,,\end{aligned}$$

whence it follows that

$$(1 + \zeta)\int_{(S)} D_2(\zeta, 0)\,\psi^{(\lambda)}(0)\,d\sigma = 0\,.$$

The last integral is an entire function of ζ; since this vanishes for $\zeta \neq -1$, it is also equal to zero for $\zeta = -1$. Hence,

$$\int_{(S)} D_2(-1, 0)\,\psi^{(\lambda)}(0)\,d\sigma = 0 \quad (\lambda = 1, 2, \ldots, k)\,. \tag{54}$$

If we now assume that $\zeta = -1$ is a pole of the function (17), then

$$D_2(-1, 0) = -\frac{1}{2\pi}\int\limits_{(S_1)} D_2(-1, 1)\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1 .$$

From Lemma 2 of §8 we conclude that

$$D_2(-1,0) = C^{(l)} \quad \text{on} \quad (S^{(l)}) \qquad (l=1, 2, \cdots, k)$$

and equals zero on $(S^{(0)})$.

Putting these values of $D_2(-1,0)$ into equation (54), we find:

$$\int\limits_{(S)} D_2(-1, 0)\,\psi^{(\lambda)}\,d\sigma = \int\limits_{(S^{(0)})} D_2(-1, 0)\,\psi^{(\lambda)}\,d\sigma + \sum_{l=1}^{k}\int\limits_{(S^{(l)})} D_2(-1, 0)\,\psi^{(\lambda)}\,d\sigma$$

$$= \sum_{l=1}^{k} C^{(l)}\int\limits_{(S^{(l)})}\psi^{(\lambda)}\,d\sigma = C^{(\lambda)} = 0 .$$

From this it follows that $D_2(-1,0)$ must vanish everywhere on (S); this however is impossible since the function $\vartheta(0)$ is irreducible. Hence, if conditions (53) are satisfied $\zeta = -1$ cannot be a pole of the function $\vartheta(0)$.

§15. The Solution of the Inner Problem for the Case (J) Assuming the Validity of Conditions (53); The Meaning of These Conditions

Even if $\zeta = -1$ is not a pole of the function $\vartheta(0)$ one is not justified in setting $\zeta = -1$ in the series (8)

$$W = W_1 + \zeta W_2 + \zeta^2 W_3 + \cdots$$

since the radius of convergence of this series may be equal to one. It is possible that W has a pole at $\zeta = 1$; however, the function

$$(1-\zeta)\,W = W_1 + \zeta(W_2 - W_1) + \zeta^2(W_3 - W_2) + \cdots$$

has no pole here. From this it follows that the value of W for $\zeta = -1$ is given by the series

$$W = \tfrac{1}{2}[\,W_1 - (W_2 - W_1) + (W_3 - W_2) - + \cdots] \tag{55}$$

and that this series provides the solution of the inner Dirichlet problem for the case (J) if conditions (53) are satisfied. These conditions are not in any direct relation to the problem under consideration, but one can easily see that they are necessary for the solution of the corresponding Problem B. Indeed, if

$$W = \int\limits_{(S_1)} \mu(1)\,\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1$$

is the potential of a double layer, then

$$\int_{(S)} W_i \psi^{(\lambda)} d\sigma = 0 \qquad (\lambda = 1, 2, \ldots, k)$$

There is therefore no potential of a double layer which fails to satisfy conditions (53). If the values of the potential on (S) are not subject to these conditions, then Problem B is not solvable.

To prove our assertion it suffices to note that

$$W_i = \left(\int_{(S_1)} \mu(1) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1\right)_i = \int_{(S_1)} \mu(1) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 + 2\pi \mu(0);$$

hence,

$$\begin{aligned}\int_{(S)} W_i \psi^{(\lambda)} d\sigma &= \int_{(S)} \psi^{(\lambda)} \left(\int_{(S_1)} \mu(1) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1\right) d\sigma + 2\pi \int_{(S)} \mu(0) \psi^{(\lambda)}(0) d\sigma \\ &= -\int_{(S_1)} \mu(1) \left(\int_{(S)} \psi^{(\lambda)} \frac{\cos(r_{01} N_1)}{r_{10}^2} d\sigma\right) d\sigma_1 + 2\pi \int_{(S)} \mu(0) \psi^{(\lambda)}(0) d\sigma \\ &= -2\pi \int_{(S_1)} \mu(1) \psi^{(\lambda)}(1) d\sigma_1 + 2\pi \int_{(S)} \mu(0) \psi^{(\lambda)}(0) d\sigma = 0.\end{aligned}$$

§16. The Solution of the Inner DIRICHLET Problem for the Case (J)

We introduce a function α of the points of (S) which is equal to zero on $(S^{(0)})$ and on $(S^{(l)})$ $(l = 1, 2, \ldots, k)$ equals a certain constant $\alpha^{(l)}$. We thus put

$$\alpha = 0 \text{ on } (S^{(0)}), \quad \alpha = \alpha^{(l)} \text{ on } (S^{(l)}) \quad (l = 1, 2, \ldots, k).$$

The numbers $\alpha^{(l)}$ are chosen such that the function $f - \alpha$ satisfies the conditions (53). Taking note of the fact that α is zero on $(S^{(0)})$, it follows that the equations

$$\int_{(S)} (f - \alpha) \psi^{(\lambda)} d\sigma = \int_{(S)} f \psi^{(\lambda)} d\sigma - \sum_{l=1}^{k} \alpha^{(l)} \int_{(S^{(l)})} \psi^{(\lambda)} d\sigma = \int_{(S)} f \psi^{(\lambda)} d\sigma - \alpha^{(\lambda)} = 0$$

must hold.

Just as in the preceding section, we now determine the solution of the inner problem corresponding to the function $f - \alpha$; we obtain:

$$w = \frac{1}{2}[w_1 - (w_2 - w_1) + (w_3 - w_2) - + \cdots], \qquad w_i = f - \alpha \tag{56}$$

with

$$w_1 = \frac{1}{2\pi}\int\limits_{(S_1)} (f-\alpha)\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots,$$

$$w_n = \frac{1}{2\pi}\int\limits_{(S_1)} \overline{w}_{n-1}\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

If the point M_0 is in the interior of (D_i), then it lies outside the regions bounded by the surfaces $(S^{(l)})$ $(l = 1, 2, \ldots, k)$; hence,

$$\begin{aligned} w_1 &= \frac{1}{2\pi}\int\limits_{(S_1)} f\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1 - \sum_{l=1}^{k}\frac{\alpha^{(l)}}{2\pi}\int\limits_{(S_1^{(l)})}\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1 \\ &= \frac{1}{2\pi}\int\limits_{(S_1)} f\frac{\cos(r_{10}N_1)}{r_{10}^2}\,d\sigma_1 = W_1. \end{aligned}$$

The same formula shows that for a point M_0 on $(S^{(\lambda)})$ the equation

$$\overline{w}_1 = \overline{W}_1 + \alpha^{(\lambda)}$$

holds, since the normals of $(S^{(\lambda)})$ in case (J) are directed into the interior of $(S^{(\lambda)})$.

It now follows that for a point M_0 on (S)

$$\overline{w}_1 = \overline{W}_1 + \alpha\,.$$

Similarly, one finds:

$$w_2 = W_2\,,\quad \overline{w}_2 = \overline{W}_2 - \alpha\,,\quad w_3 = W_3\,,\quad \overline{w}_3 = \overline{W}_3 + \alpha, \ldots.$$

This implies that for all interior points of (D_i) series (56) coincides with series (55) and that series (55) converges in the interior of (D_i) only if conditions (53) are not satisfied; in this case, however,

$$W_i = f - \alpha\,.$$

We now choose numbers $\gamma_1, \gamma_2, \ldots, \gamma_k$ such that for points on $(S^{(\lambda)})$ $(\lambda = 1, 2, \ldots, k)$ equations

$$\sum_{l=1}^{k}\gamma_l\int\limits_{(S_1)}\frac{\psi^{(l)}\,d\sigma_1}{r_{10}} = \alpha^{(\lambda)} \tag{57}$$

hold; this is always possible. Indeed, we know that the potentials

$$\int\limits_{(S_1)}\frac{\psi^{(l)}\,d\sigma_1}{r_{10}} \tag{58}$$

are constant in all regions except (D_i). If (58) equals $C^{(l)}$ inside $(S^{(\lambda)})$ $(\lambda = 1, 2, \ldots, k)$, then it follows that equations (57) are equivalent to the equations

$$\sum_{l=1}^{k} \gamma_l C_\lambda^{(l)} = \alpha^{(\lambda)}.$$

This latter system of equations has a solution for arbitrary $\alpha^{(\lambda)}$, for otherwise the system

$$\sum_{l=1}^{k} \delta_l C_\lambda^{(l)} = 0 \qquad (\lambda = 1, 2, \ldots, k)$$

would have a solution in which at least one of the numbers $\delta_1, \ldots, \delta_k$ were different from zero; then since (58) vanishes outside $(S^{(0)})$ the potential

$$\int_{(S_i)} \sum_{l=1}^{k} \delta_l \psi^{(l)} \frac{d\sigma_1}{r_{10}},$$

would vanish outside (D_i) and hence everywhere; this would imply that

$$\sum_{l=1}^{k} \delta_l \psi^{(l)} = 0,$$

which is impossible because of the linear independence of the functions $\psi^{(l)}$ $(l = 1, 2, \ldots, k)$.

We now put

$$V_0 = \int_{(S_i)} \sum_{l=1}^{k} \gamma_l \psi^{(l)} \frac{d\sigma_1}{r_{10}}$$

and consider the function

$$V = W + V_0.$$

We have:

$$V_i = W_i + (V_0)_i = f - \alpha + \alpha = f;$$

V is thus the solution of the problem in question. Just as the solution of the outer problem for the case (E), the solution of the inner problem for case (J) is also the sum of a double-layer potential and a simple-layer potential.

§17. The Outer Problem for the Case (J)

The outer problem for the case (J) is equivalent to an outer problem with the surface $(S^{(0)})$ and k inner problems with surfaces $(S^{(l)})$ $(l = 1, 2, \ldots, k)$. The formulas derived in the preceding sections therefore make it possible to immediately find the solution of this problem in the form of a single equation. One can obtain this solution by putting $\zeta = 1$ in formula (17) for the function $\vartheta(0)$ if $\zeta = 1$ is not a pole of this function.

Since $\zeta = 1$ is an eigenvalue of equation (5) and the eigenfunctions $\varrho(0)$ of equation (12) which is adjoint to (5) differ only by constant factors—we studied these eigenfunctions in III, §13—we conclude that equation (5) has a solution for $\zeta = 1$ if the equation

$$\int_{(S)} f \varrho \, d\sigma = 0 \tag{59}$$

is satisfied.

If condition (59) is fulfilled, then function (17) has no pole at $\zeta = 1$. Indeed, from the equation

$$D_2(\zeta, 0) = \frac{\zeta}{2\pi} \int_{(S_1)} D_2(\zeta, 1) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 + D_2(\zeta) f(0)$$

with the help of (59) one can derive:

$$\int_{(S)} D_2(\zeta, 0) \varrho \, d\sigma = -\frac{\zeta}{2\pi} \int_{(S_1)} D_2(\zeta, 1) \left(\int_{(S)} \varrho \frac{\cos(r_{01} N_1)}{r_{10}^2} d\sigma \right) d\sigma_{\cdot}$$

$$= \zeta \int_{(S_1)} D_2(\zeta, 1) \, \varrho(1) \, d\sigma_1 ;$$

Hence,

$$(1 - \zeta) \int_{(S)} D_2(\zeta, 0) \, \varrho(0) \, d\sigma = 0 .$$

Since the integral in the last equation is an entire function of ζ, it must vanish for $\zeta = 1$ as well as for $\zeta \neq 1$, i.e.,

$$\int_{(S)} D_2(1, 0) \, \varrho(0) \, d\sigma_{,} = 0 . \tag{60}$$

If the function $\vartheta(0)$ had a pole at $\zeta = 1$, then

$$D_2(1, 0) = \frac{1}{2\pi} \int_{(S_1)} D_2(1, 1) \frac{\cos(r_{10} N_1)}{r_{10}^2} \, d\sigma_1 .$$

From Lemma 1 of §8 this implies that

$$D_2(1,0) = C \ \text{ on } \ (S).$$

Finally, (60) then implies

$$C \int_{(S)} \varrho \, d\sigma = 0 .$$

Since the integral in the last equation is different from zero, it follows that $C = 0$ and hence that the fraction (17) is reducible; this however contradicts the hypothesis that (17) is irreducible. Thus, if condition (59) holds $\zeta = 1$ cannot be a pole of $\vartheta(0)$.

If condition (59) holds, then the function

$$(1 + \zeta) \, W = W_1 + \zeta (W_2 + W_1) + \zeta^2 (W_3 + W_2) + \cdots$$

has neither a pole at $\zeta = 1$ nor at $\zeta = -1$; the function

$$W = \tfrac{1}{2} [W_1 + (W_2 + W_1) + (W_3 + W_2) + \cdots] \tag{61}$$

provides the value of W for $\zeta = 1$, and it follows that $W_e = -f$.

Condition (59) is not directly related to the DIRICHLET problem we considered, but it is a necessary condition for the solution of the corresponding Problem B.

Any double-layer potential

$$W = \int_{(S_1)} \mu(1) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1$$

does indeed fulfill the condition

$$\int_{(S)} W_e \varrho \, d\sigma = 0 .$$

To see this, one need only repeat the considerations of §12 with but minor alterations.

We now take the eigenfunction ϱ which is defined by the equation

$$\int_{(S)} \varrho \, d\sigma = 1 ;$$

the potential

$$\int_{(S)} \frac{\varrho \, d\sigma}{r_{10}}$$

is a (nonzero) constant in the interior of (D_i). Let C be the value of this constant. If

$$\int_{(S)} f \varrho \, d\sigma = \alpha ,$$

then we form the solution w of the DIRICHLET problem corresponding to the function $f - \alpha$. Since this function satisfies condition (59), we may write:

$$w = \tfrac{1}{2}[w_1 + (w_2 + w_1) + (w_3 + w_2) + \cdots], \quad w_e = \alpha - f \qquad (61')$$

with

$$w_1 = \frac{1}{2\pi} \int_{(S_1)} (f - \alpha) \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 ,$$

$$\cdots\cdots\cdots\cdots ,$$

$$w_k = \frac{1}{2\pi} \int_{(S_1)} \overline{w}_{k-1} \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 ,$$

$$\cdots\cdots\cdots\cdots .$$

Hence,

$$w_1 = \frac{1}{2\pi} \int_{(S_1)} f \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 - \frac{\alpha}{2\pi} \int_{(S_1)} \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 = W_1 \text{ in } (D_e),$$

$$\overline{w}_1 = \frac{1}{2\pi} \int_{(S_1)} f \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 - \frac{\alpha}{2\pi} \int_{(S_1)} \frac{\cos(r_{10} N_1)}{r_{10}^2} d\sigma_1 = \overline{W}_1 - \alpha \text{ on } (S),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ;$$

from which it follows that inside (D_e) series (61′) coincides with series (61), that this series converges in (D_e), and that the equation

$$W_e = \alpha - f$$

holds. We now introduce the function

$$V_0 = \frac{\alpha}{C} \int_{(S_1)} \frac{\varrho \, d\sigma_1}{r_{10}}$$

and put

$$V = V_0 - W. \tag{62}$$

Since $$V_e = (V_0)_e - W_e = \frac{\alpha C}{C} - \alpha + f = f,$$

function (62) provides us with the solution of the problem in question.

§18. A Remark on the Question of Whether the Solution of the DIRICHLET Problem Belongs to the Class $H\ (l,A,\lambda)$

For simplicity we restrict ourselves to regions of the ordinary case and prove the following theorem:

Theorem 1. *If* $(S) \in L_{k+1}(B,\lambda)$ $(k \geqq 0)$ *and* $f \in H(l,A,\lambda)$ $(0 \leqq l \leqq k+1)$, *then the solution of the inner* (*outer*) DIRICHLET *problem belongs to the class* $H(l,cA,\lambda')$ *in* (D_i) (*in* (D_e)).

Proof. As we have seen, the solution of the inner DIRICHLET problem can always be represented as the potential of a double layer $W[\mu]$, the density μ of which satisfies the integral equation

$$\vartheta(0) = \frac{\zeta}{2\pi} \int_{(S_1)} \vartheta(1) \frac{\cos(r_{10} N_1)}{r_{10}^2} \, d\sigma_1 + \frac{f(0)}{2\pi} = \frac{\zeta}{2\pi} \overline{W}[\vartheta] + \frac{f(0)}{2\pi} \tag{63}$$

with $\zeta = -1$. We prove the theorem first for $k = 0, l = 0$. Since $f \in H(0,A,\lambda)$, $|f| < A$, and one proves just as in III, §18 that $|\vartheta| < cA$. From Theorem 1 of II, §21 it now follows that $\overline{W}[\vartheta] \in H(0,c_1A,\lambda')$. Hence on $(S)\,\vartheta \in H(0,c_2A,\lambda')$; Theorem 1 of II, §19 now implies that $W[\vartheta] \in H(0,c_3A,\lambda')$, wherewith for $l = 0, k = 0$ in the case of the inner DIRICHLET problem the theorem is proved.

We now turn to the outer problem. The solution of this problem can be represented by the potential of a double layer if and only if

$$\int_{(S_1)} \varrho_1 f \, d\sigma = 0, \tag{64}$$

where ϱ_1 is the density of a simple-layer potential which is equal to one on (S).

If condition (64) is not satisfied, then one can represent the solution of the outer DIRICHLET problem by a sum

$$W[\vartheta_1] + C\,V[\varrho_1] \tag{65}$$

in which the constant C is so chosen that the equation

$$\int_{(S)} \varrho_1 (f - C)\, d\sigma = 0$$

holds; ϑ_1 is the solution of integral equation (63) when one there puts $\zeta = 1$ and replaces f by $C - f$. Without explicitly writing this equation down, let us denote it by (63_e). Since $|f| < A$,

$$|C| = \frac{\left|\int_{(S)} \varrho_1 f\, d\sigma\right|}{\int_{(S)} \varrho_1\, d\sigma} < A,$$

for on (S) $\varrho_1 > 0$. In III, §18 with $\operatorname{Max} \varrho_1 = A_0$ it was proved that $V[\varrho_1] \in H(k + 1, cA_0, \lambda')$ in (D_e); from this it follows that $CV[\varrho_1] \in H(1, Cc_1A_0, \lambda')$ in (D_e).

We must prove that $W[\vartheta_1] \in H(0, c_2A, \lambda')$. For this we need to show that among the solutions of equation (63) there exists one which satisfies the inequality $|\vartheta_1| < c_3A$, where c_3 depends only on (S). We shall not here go into the proof of this assertion, since it is similar to the one in III, §18 for the NEUMANN problem. From the inequality $|\vartheta_1| < c_3A$ it follows that $\overline{W}[\vartheta_1] \in H(0, c_4A, \lambda')$ and from equation (63_e) that $\vartheta_1 \in H(0, c_5A, \lambda')$; hence, $W[\vartheta_1] \in H(0, c_6A, \lambda')$. Formula (65) then implies the validity of the theorem for $k = 0$, $l = 0$ in the case of the outer DIRICHLET problem.

We now come to the proof of the theorem for $k \geqq 0$, $l = 1, 2, \ldots, k + 1$. Let $f \in H(l, A, \lambda)$ on (S). It then follows from Theorem 3 of III, §19 that $W[f] \in H(l, cA, \lambda')$ in (D_i) and in (D_e). Since $l \geqq 1$, $W[f]$ has a continuous normal derivative on (S); hence, $W[f]$ can be represented as the potential of a simple layer in (D_i) as well as in (D_e). We shall denote the simple-layer potential which represents $W[f]$ in (D_i) by V_1 and that which represents it in (D_e) by V_2. The inner as well as the outer DIRICHLET problem is then solved by the simple-layer potential

$$V = \frac{1}{4\pi}(V_1 - V_2) \tag{66}$$

Indeed, on (S) the value of the potential V_1 is equal to W_i and that of the potential V_2 is equal to W_e. Hence, the potential V on (S) is given by

$$V = \frac{W_i - W_e}{4\pi} = f,$$

which proves our assertion.

To prove the theorem it suffices to show that

$$V_1 \text{ and } V_2 \in H(l, cA, \lambda') \text{ in } (D_i) \text{ and in } (D_e).$$

For V_2 the proof is simple. In (D_e)

$$W[f] \in H(l, cA, \lambda')$$

and the first derivative of $W[f] \in H(l-1,cA,\lambda')$; since $l-1 \leqq k$ it follows that $\dfrac{dW}{dn} \in H(l-1,c_1A,\lambda')$ on (S). Therefore from Theorem 1 of III, §18 $V_2 \in H(l,c_2A,\lambda')$ in (D_i) and in (D_e), for V_2 is the solution of the outer NEUMANN problem.

$$\frac{dV_{2e}}{dn} = \frac{dW}{dn}.$$

We shall now prove that also $V_1 \in H(l,cA,\lambda')$. Indeed, V_1 is a solution of the inner NEUMANN problem

$$\frac{dV_{1i}}{dn} = \frac{dW}{dn}.$$

Let V_1' be that solution of this problem which satisfies the condition

$$\int_{(S)} \varrho_1 V_1' d\sigma = 0$$

on (S). From Theorem 1 of III, §18 we can conclude that $V_1' \in H(l,c_1A,\lambda')$ in (D_i) as well as in (D_e). Clearly,

$$V_1 = V_1' + c\,V[\varrho_1]; \tag{67}$$

from this it follows that

$$V_{1i} = W_i = 2\pi f + \overline{W}[f] = V_{1i}' + c$$

and further that

$$|c| = |2\pi f + \overline{W}[f] - V_{1i}'| < c_2 A.$$

Since $V[\varrho_1] \in H(k+1,c_1A_0,\lambda')$, $V[\varrho_1] \in H(l,c_1A_0,\lambda')$; it then follows from (67) that $V_1 \in H(l,c_4A,\lambda')$, and this completes the proof of the theorem.

With the help of Theorem 3 of II, §19 and Theorem 3 of II, §21 one can prove the theorem for $l = 1, 2, \ldots, k$ in the same way as for $l = 0$.

For surfaces and functions of class $C^{(k)}$ there follows:

Theorem 2. *If $(S) \in C^{(k+1)}(B)$ $(k \geqq 1)$ and $f \in C^{(l)}(A)$ $(1 \leqq l \leqq k+1)$, then the solution of the inner (outer)* DIRICHLET *problem belongs to the class $C^{(l-1)}(cA)$ in (D_i) (in D_e)). If $f \in C^{(0)}$, then this solution belongs to the class $C^{(0)}$.*

Proof. Since $l \geqq 1$, $f \in H(l-1,c_1A,1)$ and $(S) \in L_k(c_2B,1)$; thus, from Theorem 1 we conclude that the solution belongs to the class $H(l-1,cA,\lambda)$ and hence to the class $C^{(l-1)}(cA)$. If $l = 0$, i.e., if f is continuous on (S), then the solution of the DIRICHLET problem is continuous in (D_i) and in (D_e); one cannot however assert that this solution is H-continuous. This completes the proof of the theorem.

CHAPTER V

GREEN'S FUNCTIONS AND THEIR APPLICATIONS

§1. GREEN'S Function and Its Principal Properties

Let the function $f \in H(1,A,\lambda)$ on (S). In IV, §18 we have found that it is then possible to represent the potential of a double layer $W[f]$ with density f both in (D_i) and in (D_e) by means of potentials V_1 and V_2 of simple layers which coincide with W in (D_i) and (D_e) respectively. The potential

$$V = \frac{1}{4\pi}(V_1 - V_2) \tag{1}$$

is at once the solution of the inner and outer DIRICHLET problem: $V = f$ on (S). As we have seen, it is here the case that $V \in H(1,cA,\lambda')$ in (D_i) and in (D_e).

Let $M(x,y,z)$ and $M'(\xi,\eta,\zeta)$ be two given points a distance R apart; we shall assume that the segment MM' is oriented from M to M'. We fix M in (D_i) or (D_e) and ask for the harmonic function Γ of $M'(\xi,\eta,\zeta)$ which represents the solution of the inner or outer DIRICHLET problem with the condition that

$$\Gamma = \frac{1}{R} \quad \text{on } (S). \tag{2}$$

Since the function $\frac{1}{R}$ has derivatives of arbitrary order in a neighborhood of (S), we may conclude that Γ is the potential of a simple layer which has H-continuous derivatives of first order in (D_i) or (D_e). The function Γ moreover depends on the choice of the point M and thus is also a function of $M(x,y,z)$. We put

$$G(x,y,z;\xi,\eta,\zeta) = \frac{1}{R} - \Gamma. \tag{3}$$

The function $G(x,y,z;\xi,\eta,\zeta) = G(M,M')$ is called GREEN'S *function*. Considered as a function of $M'(\xi,\eta,\zeta)$, it is harmonic at each point of (D_i) or (D_e) with the exception of the point $M(x,y,z)$ at which it becomes infinite; on (S) it is clearly equal to zero.

We shall prove that GREEN's function is a symmetric function of the points M and M', i.e.,

$$G(M, M') = G(M', M) . \tag{4}$$

We choose two distinct points M_1 and M_2 of (D_i) and consider in (D_i) the two functions

$$G_1(M') = G(M_1, M'),$$
$$G_2(M') = G(M_2, M').$$

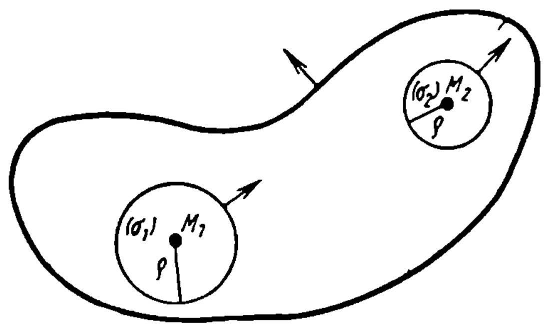

Fig. 30

Let (σ_1) and (σ_2) be the surfaces of spheres about M_1 and M_2 respectively whose radius ϱ is so small that (σ_1) and (σ_2) lie entirely in (D_i) and have no points in common (Fig. 30). We consider the subregion of (D_i) lying outside these spheres. In this region the functions $G_1(M')$ and $G_2(M')$ are harmonic and continuous and have continuous and bounded partial derivatives of first order, so that GREEN's identity is applicable.

Making use of this identity and choosing the outer normals on (σ_1) and (σ_2), we obtain:

$$\int\limits_{(S)} \left(G_1 \frac{dG_2}{dn} - G_2 \frac{dG_1}{dn}\right) d\sigma - \int\limits_{(\sigma_1)} \left(G_1 \frac{dG_2}{dn} - G_2 \frac{dG_1}{dn}\right) d\sigma$$
$$- \int\limits_{(\sigma_2)} \left(G_1 \frac{dG_2}{dn} - G_2 \frac{dG_1}{dn}\right) d\sigma = 0 .$$

Since on the surface (S)

$$G_1 = 0 , \quad G_2 = 0$$

the last equation assumes the following form:

$$\int\limits_{(\sigma_1)} \left(G_1 \frac{dG_2}{dn} - G_2 \frac{dG_1}{dn}\right) d\sigma = \int\limits_{(\sigma_2)} \left(G_2 \frac{dG_1}{dn} - G_1 \frac{dG_2}{dn}\right) d\sigma. \tag{5}$$

We study the left-hand side of this equation. On (σ_1) as well as in the interior

of the sphere bounded by (σ_1) G_2 and $\frac{dG_2}{dn}$ are bounded. Denoting the distance of the point M_1 from M' by R_1, on (σ_1) we obtain:

hence, on (σ_1)
$$\frac{d\frac{1}{R_1}}{dn} = \left(\frac{d\frac{1}{R_1}}{dR_1}\right)_{R_1=\varrho} = -\frac{1}{\varrho^2};$$

$$G_1 = \frac{1}{\varrho} - \Gamma_1, \qquad \frac{dG_1}{dn} = -\frac{1}{\varrho^2} - \frac{d\Gamma_1}{dn},$$

where Γ_1 and $\frac{d\Gamma_1}{dn}$ are bounded functions on (σ_1).

Since $G_2(M')$ has bounded derivatives of every order in a neighborhood of the point M_1, $\frac{dG_2}{dn}$ is bounded on (σ_1) and $G_2(M')$ differs from $G(M_2,M_1) = G_2(M_1)$ only by a quantity of the form $A\varrho$, where A is a bounded function. From this it follows that the integrand on the left-hand side of (5) is equal to

$$\frac{1}{\varrho^2} G(M_2, M_1) + \frac{1}{\varrho}\frac{dG_2}{dn} - \Gamma_1 \frac{dG_2}{dn} + \frac{A}{\varrho} + G(M_2, M_1)\frac{d\Gamma_1}{dn} + A\varrho\frac{d\Gamma_1}{dn}.$$

Hence, the integral on the left-hand side of equation (5) differs from $4\pi G(M_2,M_1)$ only by a quantity of order $B_1\varrho$. Similarly we conclude that the right-hand side of (5) differs from $4\pi G(M_1,M_2)$ by a quantity of order $B_2\varrho$. The difference

$$4\pi[G(M_2, M_1) - G(M_1, M_2)]$$

thus tends to zero as $\varrho \to 0$; since it does not depend on ϱ, it is therefore equal to zero. Equation (4) has now been established.

From the symmetry of $G(M,M')$ it follows that $G(M,M')$, considered as a function of the point M, is harmonic for all M' different from M. Its derivatives with respect to ξ,η,ζ and x,y,z are likewise harmonic functions at each point with

$$(\xi - x)^2 + (\eta - y)^2 + (\zeta - z)^2 > 0.$$

In a similar manner one proves the symmetry property of the GREEN's function for the region (D_e). In the following we shall denote the GREEN's function for (D_i) by $G_i(M,M')$ and that for (D_e) by $G_e(M,M')$.

Now let the point M lie in the interior of (D_i). $G_i(M,M')$ as a function of M' tends to $+\infty$ as M' approaches M; hence, on the surface (σ) of a sphere of sufficiently small radius ϱ about M the function $G_i(M,M')$ is positive. Since $G_i(M,M')$ is harmonic in the subregion of (D_i) outside this sphere and nonnegative on the boundary of the region, it is everywhere nonnegative, and we can conclude:

$$0 \leq \frac{1}{R} - \Gamma = G_i(M, M').$$

Moreover, as a harmonic function which assumes the positive value $\frac{1}{R}$ on (S), Γ is likewise positive. Therefore,

$$0 \leq G_i(M, M') = \frac{1}{R} - \Gamma < \frac{1}{R}; \tag{6}$$

the same thing can be proved for $G_e(M,M')$.

If d_0 is the diameter of the region (D_i), then

$$\int\limits_{(D_i)} \frac{1}{R^2} d\tau < 4\pi \int\limits_0^{d_0} \frac{1}{R^2} R^2 dR = 4\pi d_0 ;$$

from this we obtain the following result which will be important in what follows:

$$\int\limits_{(D_i)} [G_i(M, M')]^2 d\tau < 4\pi d_0 = A. \tag{7}$$

The integral of the square of the GREEN's function $G_i(M,M')$ is thus bounded by a number which is independent of the location of the point M. It is easily proved that the last integral is also a continuous function of M, but we shall not go into this here.

§2. The Solution of the DIRICHLET Problem for a Special Case

Let the function $F \in H(1,A,\lambda)$ on (S). The solution of the corresponding DIRICHLET problem can then be represented as the potential of a simple layer which we denote by V. If we assume that the point M lies in the interior of (D_i), then from the remarks of III, §6:

$$\begin{aligned} V &= \frac{1}{4\pi} \int\limits_{(S_1)} \frac{dV_i}{dn} \frac{d\sigma_1}{R} + \frac{1}{4\pi} \int\limits_{(S_1)} V_i \frac{\cos(RN_1)}{R^2} d\sigma_1 \\ &= \frac{1}{4\pi} \int\limits_{(S_1)} \frac{dV_i}{dn} \frac{d\sigma_1}{R} + \frac{1}{4\pi} \int\limits_{(S_1)} F \frac{\cos(RN_1)}{R^2} d\sigma_1 . \end{aligned} \tag{8}$$

Applying GREEN's identity to the functions V and Γ and taking into account the fact that $\Gamma = \frac{1}{R}$ on (S), one finds further:

$$\int\limits_{(S_1)} \left(\Gamma \frac{dV_i}{dn} - V_i \frac{d\Gamma}{dn}\right) d\sigma_1 = \int\limits_{(S_1)} \left(\frac{1}{R} \frac{dV_i}{dn} - F \frac{d\Gamma}{dn}\right) d\sigma_1 = 0.$$

Multiplying this equation by $\frac{1}{4\pi}$ and subtracting the result from (8), one obtains the relation

$$V = -\frac{1}{4\pi}\int\limits_{(S)} F\left\{\frac{d\frac{1}{R}}{dn} - \frac{d\Gamma}{dn}\right\} d\sigma = -\frac{1}{4\pi}\int\limits_{(S)} F\frac{dG_i}{dn}\, d\sigma. \tag{9}$$

Equation (9) makes it possible to solve the DIRICHLET problem for any function F satisfying the above conditions if the GREEN's function is known. We have thus reduced the general DIRICHLET problem to a very special problem, namely, finding the function Γ.

The computations above were carried out for the inner problem. For the outer problem we need only change the sign in the identity (8); we thus obtain for the solution of the outer problem the following formula:

$$V = \frac{1}{4\pi}\int\limits_{(S)} F\frac{dG_e}{dn}\, d\sigma. \tag{9'}$$

Scholium. Suppose that $F = 1$. Then for the inner problem $V \equiv 1$, and in (D_i) the equation

$$-\int\limits_{(S)}\frac{dG_i}{dn}\, d\sigma = 4\pi$$

holds. In (D_e) this equation is no longer valid. However, the right-hand side of (9′) tends to one as the point M approaches the surfaces (S) from the outer side. If this point is sufficiently close to (S), then the difference

$$\int\limits_{(S)}\frac{dG_e}{dn}\, d\sigma - 4\pi$$

can be made arbitrarily small. The quantity

$$\frac{1}{4\pi}\int\limits_{(S)}\frac{dG_e}{dn}\, d\sigma$$

is a harmonic function which assumes the value one on (S) and vanishes at infinity; at each point of (D_e) it is therefore greater than zero and less than one.

§3. A Lemma Due to LYAPUNOV

For subsequent considerations we need an inequality due to LYAPUNOV which we shall now derive. Let $M_0(x_0,y_0,z_0)$ be a point of the surface (S). We wish to solve an inner DIRICHLET problem by means of the method presented in §2; let the function F be

$$F = (\xi - x_0)^2 + (\eta - y_0)^2 + (\zeta - z_0)^2.$$

this point from M_0. The second factor is nonnegative, and the integral is therefore no smaller than

$$\frac{\varrho^2}{4\pi}\int\limits_{(S-\sigma)}\left(-\frac{dG_i}{dn}\right)d\sigma.$$

Hence,

$$\frac{\varrho^2}{4\pi}\int\limits_{(S-\sigma)}\left(-\frac{dG_i}{dn}\right)d\sigma \leq U < A\,\delta^\lambda$$

and therefore

$$\frac{1}{4\pi}\int\limits_{(S-\sigma)}\left(-\frac{dG_i}{dn}\right)d\sigma < \frac{A\,\delta^\lambda}{\varrho^2}. \tag{13}$$

This inequality is the content of the lemma due to LYAPUNOV which we wished to prove.

If the point lies in the interior of (D_e), then in formula (10) one must change the sign; in this case G increases from zero to $+\infty$ as the point M' of (S) approaches M, and $\frac{dG_e}{dn}$ is positive. Hence, for points in the interior of (D_e) we have the inequality

$$\frac{1}{4\pi}\int\limits_{(S-\sigma)}\frac{dG_e}{dn}\,d\sigma < \frac{A\,\delta^\lambda}{\varrho^2}. \tag{13'}$$

§4. The Solution of the DIRICHLET Problem in the General Case

Let f be a function on (S); it is required to find the harmonic function defined in the interior of (D_i) which assumes the same values as f on (S). We shall prove that the solution of this problem is given by the following function:

$$V = -\frac{1}{4\pi}\int\limits_{(S)} f\,\frac{dG_i}{dn}\,d\sigma \tag{14}$$

We note first of all that the function V is harmonic in (D_i), for

$$V = -\frac{1}{4\pi}\int\limits_{(S_1)} f\left\{\frac{\partial G_i}{\partial\xi}\cos(N_1 x) + \frac{\partial G_i}{\partial\eta}\cos(N_1 y) + \frac{\partial G_i}{\partial\zeta}\cos(N_1 z)\right\}d\sigma_1;$$

the quantities

$$\frac{\partial G_i}{\partial\xi}, \qquad \frac{\partial G_i}{\partial\eta}, \qquad \frac{\partial G_i}{\partial\zeta}$$

considered as functions of $M(x,y,z)$, are harmonic in the interior of (D_i).

Let the point M lie on the normal to (S) at M_0 at a distance δ from M_0. We consider the cylinder introduced in the preceding section. If we denote the value of f at M_0 by f_0 and recall the remarks made at the end of §2, we find:

$$
\begin{aligned}
V - f_0 &= -\frac{1}{4\pi}\int\limits_{(S)} f \frac{dG_i}{dn}\, d\sigma - f_0 \\
&= -\frac{1}{4\pi}\int\limits_{(S)} f \frac{dG_i}{dn}\, d\sigma - \left(-\frac{1}{4\pi}\int\limits_{(S)} f_0 \frac{dG_i}{dn}\, d\sigma\right) \qquad (15)\\
&= \frac{1}{4\pi}\int\limits_{(S)} (f - f_0)\left(-\frac{dG_i}{dn}\right) d\sigma .
\end{aligned}
$$

We write this equation in the form

$$V - f_0 = \frac{1}{4\pi}\int\limits_{(\sigma)} (f - f_0)\left(-\frac{dG_i}{dn}\right) d\sigma + \frac{1}{4\pi}\int\limits_{(S-\sigma)} (f - f_0)\left(-\frac{dG_i}{dn}\right) d\sigma .$$

If we assume that f is less than B on (S), then

$$|f - f_0| < 2B .$$

Using the continuity of the function f we choose ϱ so small that

$$|f - f_0| < \varepsilon \quad \text{on} \quad (\sigma) .$$

Then

$$
\begin{aligned}
&\left|\frac{1}{4\pi}\int\limits_{(\sigma)} (f - f_0)\left(-\frac{dG_i}{dn}\right) d\sigma\right| \\
&\qquad \le \frac{1}{4\pi}\int\limits_{(\sigma)} |f - f_0| \left(-\frac{dG_i}{dn}\right) d\sigma < \frac{\varepsilon}{4\pi}\int\limits_{(\sigma)} \left(-\frac{dG_i}{dn}\right) d\sigma \\
&\qquad \le \frac{\varepsilon}{4\pi}\int\limits_{(S)} \left(-\frac{dG_i}{dn}\right) d\sigma = \varepsilon .
\end{aligned}
$$

For the second integral we have the estimate

$$\frac{1}{4\pi}\left|\int\limits_{(S-\sigma)} (f - f_0)\left(-\frac{dG_i}{dn}\right) d\sigma\right| < \frac{2B}{4\pi}\int\limits_{(S-\sigma)} \left(-\frac{dG_i}{dn}\right) d\sigma < \frac{2BA\delta^\lambda}{\varrho^2} ;$$

hence,

$$|V - f_0| < \varepsilon + \frac{2AB\delta^\lambda}{\varrho^2} . \qquad (16)$$

We now choose a number δ_0 such that the second summand in (16) is less than ε for $\delta = \delta_0$. We then find:

$$|V - f_0| < 2\varepsilon , \qquad \text{if} \qquad \delta < \delta_0 .$$

From this we see that $\lim V = f_0$ as M approaches M_0 along the normal to (S) through M_0.

By again repeating arguments already used many times, we can show that it is always the case that $\lim V = f_0$ if M approaches M_0 while always remaining

on the same side of (S) (Fig. 32). For if M_1 is the point of (S) nearest to M, then

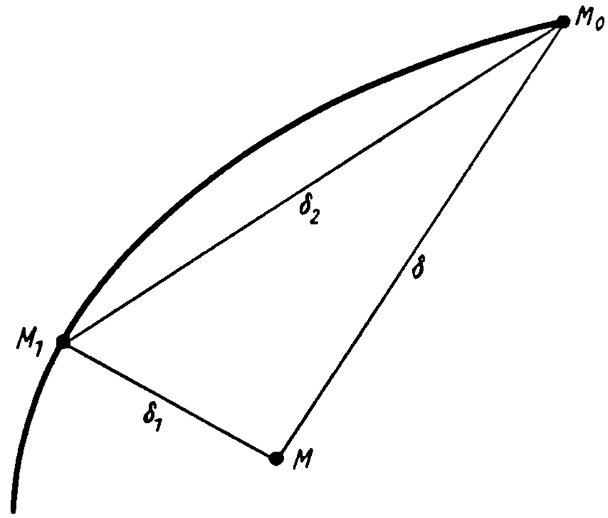

Fig. 32

M lies on the normal to (S) at M_1; if δ_1 and δ_2 are the distances of the point M_1 from the points M and M_0 respectively, then the inequalities

$$\delta_1 < \delta, \quad \delta_2 < \delta_1 + \delta < 2\delta$$

hold. Therefore, for sufficiently small δ

$$|f_{M_1} - f_{M_0}| < \varepsilon, \quad |V_M - f_{M_1}| < \varepsilon;$$

from this it follows that

$$|V_M - f_{M_0}| \leq |V_M - f_{M_1}| + |f_{M_1} - f_{M_0}| < 2\varepsilon.$$

In solving the outer DIRICHLET problem it is necessary to change the sign in formula (14) and to alter formula (15) somewhat. In this case f_0 is not equal to

$$\frac{1}{4\pi} \int_{(S)} f_0 \frac{dG_e}{dn} \, d\sigma,$$

but differs from this expression by a quantity which is less than ε in absolute value if δ is less than a certain $\delta^{(0)}$. Inequality (16) is then replaced by the inequality

$$|V - f_0| < 2\varepsilon + \frac{2AB\delta^\lambda}{\varrho^2}, \qquad \text{if} \quad \delta < \delta^{(1)}, \tag{16'}$$

from which the same conclusions follow as from (16).

§5. The Function of F. NEUMANN and Its Properties

In the outer NEUMANN problem for the ordinary case and case (E)—we exclude the outer problem for the case (J)—it is always possible to find the function which is harmonic in the interior of (D_e) and has the normal derivatives

$$\frac{\cos(R\,N)}{R^2} \tag{17}$$

on (S).

The inner problem for the ordinary case and case (J)—we exclude the inner problem for the case (E)—has no solution under such conditions. Indeed,

$$\int\limits_{(S_1)} \frac{\cos(R\,N_1)}{R^2}\, d\sigma_1 = 4\pi,$$

while for the solvability of this problem it is necessary that this integral vanish. For this reason, when considering the inner problem we replace the function (17) by the function

$$\frac{\cos(R\,N)}{R^2} - \frac{4\pi}{S}, \tag{17'}$$

wherein S denotes the area of the entire boundary; the problem is then solvable. Let $\Gamma^{(i)}$ be one of the harmonic functions which solves this problem; all other solutions are given by the formula

$$\Gamma^{(i)} + C(M)\;;$$

where $C(M)$ is an arbitrary function of M.

For the outer problem we put

$$G^{(e)}(x, y, z;\, \xi, \eta, \zeta) = \frac{1}{R} + \Gamma^{(e)} \tag{18}$$

and for the inner problem

$$G^{(i)}(x, y, z;\, \xi, \eta, \zeta) = \frac{1}{R} + \Gamma^{(i)} + C(M); \tag{18'}$$

in the latter case we choose the quantity $C(M)$ such that the condition

$$\int\limits_{(S)} G^{(i)}(x, y, z;\, \xi, \eta, \zeta)\, d\sigma \equiv 0 \quad \text{for all } M \text{ in } (D_i) \tag{19}$$

is satisfied.

The functions so defined are called NEUMANN *functions*. They play the same role in the NEUMANN problem as do the GREEN's functions in the DIRICHLET problem.

Each NEUMANN function in its region of definition is a harmonic function of the point M' except at the point $\xi = x, \eta = y, \zeta = z$ where it becomes infinite.

From the relation

$$\frac{d\frac{1}{R}}{dn} = -\frac{\cos(RN)}{R^2}$$

we obtain for the function $G^{(e)}$:

$$\frac{dG_e^{(e)}}{dn} = \frac{d\Gamma_e^{(e)}}{dn} + \frac{d\frac{1}{R}}{dn} = \frac{\cos(RN)}{R^2} - \frac{\cos(RN)}{R^2} = 0 \quad \text{on} \quad (S).$$

The function $G^{(i)}$ has the following property:

$$\frac{dG_i^{(i)}}{dn} = \frac{d\Gamma_i^{(i)}}{dn} + \frac{d\frac{1}{R}}{dn} = \frac{\cos(RN)}{R^2} - \frac{4\pi}{S} - \frac{\cos(RN)}{R^2} = -\frac{4\pi}{S} \quad \text{on} \quad (S).$$

One can easily prove that $G^{(i)}$ and $G^{(e)}$ are symmetric functions of the points M and M', i.e.,

$$G(M, M') = G(M' M)$$

where

$$G(x, y, z;\ \xi, \eta, \zeta) = G(M, M').$$

We shall study the function $G^{(i)}$. Just as in §1, we consider spheres of radius ϱ about the points M_1 and M_2 with surfaces (σ_1) and (σ_2) and direct our attention to the region bounded by the surfaces (S), (σ_1), and (σ_2).

GREEN's identity gives:

$$\int\limits_{(S)}\left(G_1^{(i)}\frac{dG_{2i}^{(i)}}{dn} - G_2^{(i)}\frac{dG_{1i}^{(i)}}{dn}\right)d\sigma - \int\limits_{(\sigma_1)}\left(G_1^{(i)}\frac{dG_{2i}^{(i)}}{dn} - G_2^{(i)}\frac{dG_{1i}^{(i)}}{dn}\right)d\sigma$$
$$- \int\limits_{(\sigma_2)}\left(G_1^{(i)}\frac{dG_{2i}^{(i)}}{dn} - G_2^{(i)}\frac{dG_{1i}^{(i)}}{dn}\right)d\sigma = 0$$

with

$$G_1^{(i)} = G^{(i)}(M_1, M'), \qquad G_2^{(i)} = G^{(i)}(M_2, M').$$

Each of the normal derivatives $\frac{dG_{1i}^{(i)}}{dn}$ and $\frac{dG_{2i}^{(i)}}{dn}$ is equal to $-\frac{4\pi}{S}$ on the surface (S). Hence, the first integral of the last identity equals

$$-\frac{4\pi}{S}\int\limits_{(S)}(G_1^{(i)} - G_2^{(i)})\,d\sigma,$$

and so from (19) it is equal to zero. Our identity now takes the following form:

$$\int\limits_{(\sigma_1)} \left(G_1^{(i)} \frac{d G_{2i}^{(i)}}{d n} - G_2^{(i)} \frac{d G_{1i}^{(i)}}{d n} \right) d\sigma = \int\limits_{(\sigma_2)} \left(G_2^{(i)} \frac{d G_{1i}^{(i)}}{d n} - G_1^{(i)} \frac{d G_{2i}^{(i)}}{d n} \right) d\sigma . \tag{20}$$

This equation has the same form as equation (5) in §1; by exactly repeating the arguments of that section, we come to the conclusion that

$$G^{(i)}(M_1, M_2) = G^{(i)}(M_2, M_1).$$

For the outer problem one finds immediately:

$$\frac{d G_{1e}^{(e)}}{d n} = 0, \qquad \frac{d G_{2e}^{(e)}}{d n} = 0 \qquad \text{on } (S)$$

this leads directly to equation (20).

Remark. We excluded the inner NEUMANN problem for the case (E) and the outer problem for the case (J) only in order to not encumber the presentation. Making use of the theorems of Chapter III, one can construct NEUMANN functions which are also valid for the excluded cases. It suffices in (17′) to choose the additional terms corresponding to the boundaries $(S^{(l)})$ $(l = 1, 2, \ldots, k)$ such that the conditions for the solvability of the problem are satisfied and to add a term to the solution found which is independent of ξ, η, ζ and is so chosen that condition (19) is satisfied for each of the boundary surfaces $(S^{(l)})$ $(l = 1, 2, \ldots, k)$. The rest of the considerations remain valid in the excluded cases.

In §1 we obtained estimates (6) and (7) for the GREEN's function. We wish to show that for the NEUMANN function $G^{(i)}(M,M')$ the inequality

$$\int\limits_{(D_i)} [G^{(i)}(M, M')]^2 \, d\tau < A$$

holds.

We first of all prove that there exists a constant $c > 0$ such that

$$\int\limits_{(S)} |\mu| \, d\sigma \leq c \int\limits_{(S)} |f| \, d\sigma \tag{21}$$

where μ is the density of a simple-layer potential which solves the inner NEUMANN problem

$$\frac{d V_i}{d n} = f \; ;$$

μ must satisfy the condition

$$\int\limits_{(S)} \mu \, d\sigma = 0 .$$

Indeed, as we have seen in Chapter III, the density μ is determined by the formula

$$\mu = \sum\nolimits_n (0) + \int\limits_{(S_1)} B_n(1, 0) \sum\nolimits_n (1) \, d\sigma_1 ;$$

from this it follows that

$$\int_{(S)} |\mu|\, d\sigma \le \int_{(S_0)} \left|\sum\nolimits_n(0)\right| d\sigma_0 + \int_{(S_1)} \left|\sum\nolimits_n(1)\right| \left[\int_{(S_0)} |B_n(1,0)|\, d\sigma_0\right] d\sigma_1$$
$$\le c' \int_{(S)} \left|\sum\nolimits_n(0)\right| d\sigma_0 .$$

It is easily seen that the inequality

$$\int_{(S)} \left|\sum\nolimits_n(0)\right| d\sigma_0 \le c'' \int_{(S)} |f|\, d\sigma$$

holds, for

$$\int_{(S_0)} \left|\int_{(S_1)} K(1,0) f(1)\, d\sigma_1\right| d\sigma_0 \le \int_{(S_0)} \left[\int_{(S_1)} |K(1,0)| \cdot |f(1)|\, d\sigma_1\right] d\sigma_0$$
$$= \int_{(S_0)} |f(0)| \left[\int_{(S_1)} |K(0,1)|\, d\sigma_1\right] d\sigma_0 \le c''' \int_{(S_0)} |f(0)|\, d\sigma_0 .$$

Analogous inequalities are obtained for all the summands of the sum $\Sigma_n(0)$. This proves the inequality (21).

Let (ω) be an arbitrary region of diameter δ which together with its boundary lies in (D_i). One can then estimate the integral of the square of $V[\mu]$ over (ω) in the following manner:

$$\int_{(\omega)} |V[\mu]|^2\, d\tau = \int_{(\omega)} \left[\int_{(S_1)} \frac{\mu(1)}{r_{10}}\, d\sigma_1\right]\left[\int_{(S_2)} \frac{\mu(2)}{r_{20}}\, d\sigma_2\right] d\tau$$
$$= \int_{(S_1)} \mu(1) \left[\int_{(S_2)} \mu(2) \left(\int_{(\omega)} \frac{d\tau}{r_{10} r_{20}}\right) d\sigma_2\right] d\sigma_1$$
$$\le 4\pi\delta \left[\int_{(S)} |\mu(1)|\, d\sigma_1\right]^2 \le 4\pi\delta c^2 \left[\int_{(S)} |f|\, d\sigma\right]^2;$$

here

$$\int_{(\omega)} \frac{d\tau}{r_{10} r_{20}} \le \sqrt{\int_{(\omega)} \frac{d\tau}{r_{10}^2} \int_{(\omega)} \frac{d\tau}{r_{20}^2}} < \sqrt{\left(4\pi \int_0^\delta \frac{r^2\, dr}{r^2}\right)^2} = 4\pi\delta .$$

Hence, denoting by d_0 the diameter of the region (D_i),

$$\int_{(D_i)} |V[\mu]|^2\, d\tau \le 4\pi d_0 c^2 \left[\int_{(S)} |f|\, d\sigma\right]^2 = c_2 \left[\int_{(S)} |f|\, d\sigma\right]^2. \tag{22}$$

One sees similarly that the inequality

$$\int_{(S)} |V[\mu]|\, d\sigma \le c_3 \int_{(S)} |f|\, d\sigma \tag{23}$$

holds.

The function $\Gamma^{(i)}(M,M')$ is defined as the solution of the inner NEUMANN problem with

$$f = -\frac{4\pi}{S} + \frac{\cos(RN)}{R^2};$$

we have:

$$\int\limits_{(S)} |f|\, d\sigma \leq 4\pi + \int\limits_{(S)} \frac{|\cos(RN)|}{R^2}\, d\sigma .$$

We wish to prove that the integral on the right-hand side is bounded. If the distance of the point M from (S) is no less than $\frac{d}{2}$ (d is the radius of the LYAPUNOV sphere), then the integral is not greater than $\frac{4}{d^2}S$. Let now the distance δ of the point M from the surface (S) be less than $\frac{d}{2}$, and let M_0 be the point of (S) nearest M. As usual, we denote by (Σ) and (σ) the subregions of (S) inside the LYAPUNOV sphere and inside the sphere of radius 2δ about M_0. The distance of the point M from the surface $(S - \Sigma)$ is then not less than $\frac{d}{2}$, and the integral over $(S - \Sigma)$ is no greater than $\frac{4}{d^2}S$. The surface (σ) is not larger than $2\pi(2\delta)^2 = 8\pi\delta^2$, while R is not less than δ. Hence, the integral over (σ) is no greater than 8π.

It remains to consider the integral over $(\Sigma - \sigma)$. We denote the distance of the point of integration M' from M_0 by R and assume that the segments $M'M$ and $M'M_0$ are oriented from M' to M and M_0 respectively; then

$$|R\cos(RN) - R_0\cos(R_0N)| = |\delta\cos(\delta, N)| \leq \delta$$

and hence

$$\frac{|\cos(RN)|}{R^2} \leq \frac{\delta}{R_3} + \frac{R_0}{R^3}|\cos(R_0N)| < \frac{\delta}{\varrho^3} + \frac{2\varrho\cdot E(2\varrho)^\lambda}{\varrho^3} = \frac{\delta}{\varrho^3} + 2^{\lambda+1}E\varrho^{\lambda-2}.$$

From this there follows the estimate

$$\int\limits_{(\Sigma-\sigma)} \frac{|\cos(RN)|}{R^2}\, d\sigma < 2\cdot 2\pi\delta\int\limits_{\delta}^{d} \frac{\varrho}{\varrho^3}\, d\varrho + 2^{\lambda+3}\pi E\int\limits_{\delta}^{d} \varrho^{\lambda-2}\varrho\, d\varrho$$

$$= 4\pi\left(1 - \frac{\delta}{d}\right) + \frac{2^{\lambda+3}\pi E}{\lambda}(d^\lambda - \delta^\lambda) < 4\pi + \frac{2^{\lambda+3}\pi E}{\lambda}d^\lambda .$$

This proves that the integral of $|f|$ is bounded.

From inequalities (22) and (23) it follows that

$$\int\limits_{(D_i)} |\Gamma^{(i)}(M, M')|^2\, d\tau < A, \tag{24}$$

$$\int\limits_{(S)} |\Gamma^{(i)}(M, M')|\, d\sigma < B. \tag{25}$$

Integrating (18′) over (S) and making use of condition (19), we find:

$$|C(M)| = \frac{1}{S}\left|\int\limits_{(S)} \frac{1}{R}\, d\sigma + \int\limits_{(S)} \Gamma^{(i)}\, d\sigma\right| < \frac{1}{S}\left[B + \int\limits_{(S)} \frac{1}{R}\, d\sigma\right] < C\,.$$

Since the integrals

$$\int\limits_{(D_i)} \frac{1}{R^2}\, d\tau\,, \qquad \int\limits_{(D_i)} |\Gamma^{(i)}(M, M')|^2\, d\tau\,, \qquad \int\limits_{(D_i)} [C(M)]^2\, d\tau$$

are bounded by numbers which do not depend on the position of the point M, the integral of the square of $G^{(i)}(M,M')$ is bounded, as was to be proved. Moreover,

$$\int\limits_{(\omega)} |G^{(i)}(M, M')|^2\, d\tau < c\,\delta\,,$$

where δ is the diameter of the region (ω) lying in (D_i) which contains the point M.

§6. The Solution of the NEUMANN Problem

We shall first of all concern ourselves with the inner problem. Let f be a continuous function given on (S); it is required to find a function harmonic in (D_i) with normal derivative equal to f. We know that this problem can be solved by the potential of a simple layer; let V be this potential.

From GREEN's formulas we may write:

$$\begin{aligned} V &= \frac{1}{4\pi}\int\limits_{(S)} V_i \frac{\cos(RN)}{R^2}\, d\sigma + \frac{1}{4\pi}\int\limits_{(S)} \frac{dV_i}{dn}\cdot\frac{d\sigma}{R} \\ &= \frac{1}{4\pi}\int\limits_{(S)} V_i \frac{\cos(RN)}{R^2}\, d\sigma + \frac{1}{4\pi}\int\limits_{(S)} f\,\frac{d\sigma}{R}\,. \end{aligned} \tag{26}$$

Here, just as in §1, R is the distance from $M(x,y,z)$ to $M'(\xi,\eta,\zeta)$; the functions V_i and f will be considered as functions of ξ, η, ζ.

GREEN's identity gives:

$$\int\limits_{(S)}\left(V_i \frac{d\Gamma_i^{(i)}}{dn} - \Gamma_i^{(i)} \frac{dV_i}{dn}\right) d\sigma = \int\limits_{(S)}\left(V_i \frac{d\Gamma_i^{(i)}}{dn} - \Gamma_i^{(i)} f\right) d\sigma = 0\,.$$

Now on the surface (S)

$$\frac{d\Gamma_i^{(i)}}{dn} = \frac{\cos(RN)}{R^2} - \frac{4\pi}{S}\,;$$

the last identity therefore takes the form

$$\int_{(S)} V_i \frac{\cos(RN)}{R^2}\, d\sigma - \int_{(S)} \Gamma_i^{(i)} f\, d\sigma - \frac{4\pi}{S}\int_{(S)} V_i\, d\sigma = 0\,,$$

or, what is the same thing, the form

$$\int_{(S)} V_i \frac{\cos(RN)}{R^2}\, d\sigma - \int_{(S)} \Gamma_i^{(i)} f\, d\sigma - c = 0\,; \tag{27}$$

c is here a constant equal to

$$\frac{4\pi}{S}\int_{(S)} V_i\, d\sigma\,.$$

If we multiply equation (27) by $\dfrac{1}{4\pi}$ and subtract it from (26), we obtain finally:

$$V = \frac{1}{4\pi}\int_{(S)} f\left\{\frac{1}{R} + \Gamma_i^{(i)}\right\} d\sigma + c' = \frac{1}{4\pi}\int_{(S)} f G^{(i)}\, d\sigma + c'\,.$$

After we have solved the NEUMANN problem in a special case in order to determine $G^{(i)}$, we now also obtain the solution of the general problem with the help of the formula

$$V = \frac{1}{4\pi}\int_{(S)} f G^{(i)}\, d\sigma + c'\,,$$

where c' is a constant.

In the case of the outer problem the solution is simpler. Changing the sign in (26), repeating our arguments, and noting that

$$\frac{d\Gamma_e^{(e)}}{dn} = \frac{\cos(RN)}{R^2}\,,$$

we obtain the solution of the outer problem in the form

$$V = -\frac{1}{4\pi}\int_{(S)} f G^{(e)}\, d\sigma\,.$$

§7. The Problem of Stationary Temperature

We suppose that heat is distributed in a body (D) in such a manner that the temperature at each point of (D) remains constant in time; this is only possible if the temperature on the surface of the body is held constant.

Let $d\sigma$ be an element of a surface either in the interior of the body or on its boundary; $d\sigma$ may be taken to be planar. An upright cylinder is now constructed whose generators have the same direction as the normal of $d\sigma$. We assume that this normal is directed from the base to the top and denote by u_1 the temperature of the lower surface and by u_2 the temperature of the top surface (Fig. 33).

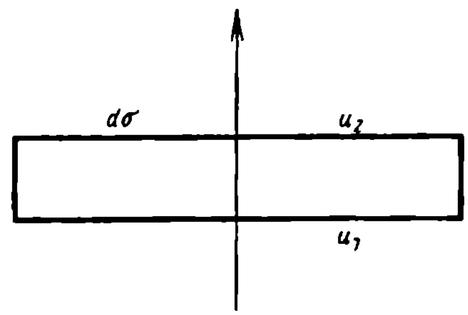

Fig. 33

The amount of heat which in time dt passes through the cylinder in the direction of the normal is proportional to the difference $u_1 - u_2$, the surface element $d\sigma$, and the time dt; it is inversely proportional to the height of the cylinder. Hence,

$$\Delta q = k \frac{u_1 - u_2}{\Delta n} d\sigma\, dt \quad (k > 0).$$

Letting Δn tend to zero, we find for the amount of heat which in time dt passes through the surface element $d\sigma$:

$$dq = -k \frac{du}{dn} d\sigma\, dt;$$

u is here the temperature, and k is a factor of proportionality which is called the *thermal conductivity*. We shall suppose that the body is homogeneous so that k is a constant. The amount of heat which passes from the interior of a region (ω) in the body (D) in time dt through the surface (σ) of (ω) is equal to

$$-dt \cdot k \int\limits_{(\sigma)} \frac{du}{dn} d\sigma.$$

Since no heat exchange takes place between individual parts of the body, this last quantity must vanish; thus

$$\int\limits_{(\sigma)} \frac{du}{dn} d\sigma = \int\limits_{(\omega)} \Delta u\, d\tau = 0.$$

Since the region (ω) was arbitrary, it now follows that

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{28}$$

The amount of heat which passes through an element $d\sigma$ of the surface of (D) is proportional to the temperature difference between points of the surface and points of the surrounding space; we here assume that the surrounding space has a constant temperature u_0. Hence, on the outer surface

$$-k \frac{du_i}{dn} d\sigma\, dt = [\lambda(u_i - u_0) + f']\, d\sigma\, dt \quad (\lambda > 0);$$

λ is called the *radiation coefficient.* Thus, the temperature on the surface is subject to the condition

$$\frac{d u_i}{d n} = -h(u_i - u_0) + f, \tag{29}$$

where h is a positive constant.

If for simplicity we put

$$V = u - u_0,$$

then we see that the problem reduces to finding a function V which is the solution of the LAPLACE equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \tag{28'}$$

with boundary condition

$$\frac{d V_i}{d n} = -h V_i + f. \tag{29'}$$

We shall determine V in the form of a simple-layer potential and therefore put

$$V = \frac{1}{2\pi} \int_{(S_1)} \frac{\varrho(1)}{r_{10}} \, d\sigma_1 . \tag{30}$$

The function V satisfies equation (28′). Since

$$\frac{d V_i}{d n} = \frac{1}{2\pi} \int_{(S_1)} \frac{\varrho(1) \cos(r_{10} N_0)}{r_{10}^2} \, d\sigma_1 + \varrho(0) ,$$

we find from condition (29′) that

$$\varrho(0) = -\frac{1}{2\pi} \int_{(S_1)} \varrho(1) \left\{ \frac{\cos(r_{10} N_0)}{r_{10}^2} + \frac{h}{r_{10}} \right\} d\sigma_1 + f(0);$$

the density $\varrho(0)$ therefore satisfies the integral equation

$$\varrho(0) = \frac{\lambda}{2\pi} \int_{(S_1)} \varrho(1) \left\{ \frac{\cos(r_{10} N_0)}{r_{10}^2} + \frac{h}{r_{10}} \right\} d\sigma_1 + f(0) \tag{31}$$

with $\lambda = -1$.

The kernel of equation (31) is not bounded; it is nevertheless clear that one of its iterated kernels is bounded, for one can apply the considerations of III, §4 to the kernel. It is easy to show that $\lambda = -1$ is not an eigenvalue of equation (31), or in other words that the problem at hand has exactly one solution. If $\lambda = -1$ were an eigenvalue, then the equation

$$\varrho(0) = -\frac{1}{2\pi} \int_{(S_1)} \varrho(1) \left\{ \frac{\cos(r_{10} N_0)}{r_{10}^2} + \frac{h}{r_{10}} \right\} d\sigma_1$$

would have a nonzero solution, and the function V would satisfy the boundary condition

$$\frac{dV_i}{dn} = -hV.$$

Multiplying this equation by V and integrating over (S), we find:

$$-\int\limits_{(S)} hV^2\,d\sigma = \int\limits_{(S)} V\frac{dV_i}{dn}\,d\sigma = \int\limits_{(D)}\left\{\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2\right\}d\tau.$$

The last equation however can hold only if $V \equiv 0$, for if not the two outer terms would have different signs.

It has now been shown that the problem has a unique solution. We note that all considerations remain valid if it is assumed that h is an arbitrary positive function continuous on (S).

After having established the existence of a unique solution, we wish to actually construct this solution. We return to equation (28′) and put

$$V = v + \frac{C}{h},$$

where C is a constant. Putting this into equations (28′) and (29′), we find:

$$\Delta v = 0 \text{ in } (D_i), \qquad \frac{dv_i}{dn} + hv_i = f - C \quad \text{on } (S). \tag{32}$$

We choose C such that the equation

$$\int\limits_{(S)} (f - C)\,d\sigma = 0 \tag{33}$$

holds and put

$$v = v_0 + hv_1 + h^2 v_2 + \cdots. \tag{34}$$

Substituting this expression for v into (32) and comparing coefficients of powers of h, one arrives at a system of equations which makes it possible to compute the functions $v_0, v_1, \ldots$ in succession:

$$\left.\begin{array}{l} \Delta v_0 = 0 \\ \Delta v_1 = 0 \\ \Delta v_2 = 0 \\ \cdots\cdots \end{array}\right\} \text{ in } (D_i), \qquad \left.\begin{array}{l} \dfrac{dv_0}{dn} = f - C \\ \dfrac{dv_1}{dn} = -v_0 \\ \dfrac{dv_2}{dn} = -v_1 \\ \cdots\cdots\cdots \end{array}\right\} \text{ on } (S).$$

Since condition (33) holds, there exists a function v_0; having found a solution v_0, one adjoins to it a constant c_0 such that the equation

$$\int\limits_{(S)} v_0\,d\sigma = 0$$

holds.

One can now determine v_1 and add to it a constant such that

$$\int_{(S)} v_1 \, d\sigma = 0 \; ;$$

one now computes v_2 etc.

It remains to learn if the series (34) converges. Using the NEUMANN function, we find successively:

$$v_0 = \frac{1}{4\pi} \int_{(S_1)} (f - C) \, G^{(i)} \, d\sigma_1 + c_0 ,$$

$$v_1 = -\frac{1}{4\pi} \int_{(S_1)} v_0 G^{(i)} \, d\sigma_1 + c_1 ,$$

$$v_2 = -\frac{1}{4\pi} \int_{(S_1)} v_2 G^{(i)} \, d\sigma_1 + c_2 ,$$

. . . .

Let M_k be the maximum $| v_k |$ on (S). One finds easily:

$$\left| \frac{1}{4\pi} \int_{(S_1)} v_k G^{(i)} \, d\sigma_1 \right| \leq M_k \frac{1}{4\pi} \int_{(S_1)} | G^{(i)} | \, d\sigma_1 = M_k N \; ; \tag{35}$$

here

$$N = \frac{1}{4\pi} \int_{(S_1)} | G^{(i)} | \, d\sigma_1 \leq \frac{1}{4\pi} \int_{(S_1)} \frac{1}{R} \, d\sigma_1 + \frac{1}{4\pi} \int_{(S_1)} | \Gamma_i | \, d\sigma_1 + \frac{1}{4\pi} | C(M) | \, S$$

$$< \frac{2}{4\pi} \int_{(S)} \frac{1}{R} \, d\sigma + \frac{2B}{4\pi} \leq \frac{1}{2\pi} \left(\operatorname{Max} \int_{(S)} \frac{1}{R} \, d\sigma + B \right) = N_0 .$$

We have here made use of inequality (25) of §5 and the inequality following it.

We shall now return to inequality (35); from the equation

$$\int_{(S)} v_{k+1} \, d\sigma = \int_{(S)} \left(-\frac{1}{4\pi} \int_{(S_1)} v_k G^{(i)} \, d\sigma_1 \right) d\sigma + c_{k+1} S = 0$$

it follows that

$$| c_{k+1} | \, S = \left| \int_{(S)} \left(\frac{1}{4\pi} \int_{(S_1)} v_k \, G^{(i)} \, d\sigma_1 \right) d\sigma \right| \leq M_k N_0 S \; ;$$

for

$$| c_{k+1} | \leq M_k N_0 , \qquad | v_{k+1} | \leq 2 N_0 M_k .$$

One thus finds easily:

$$| v_{k+1} | \leq 2 N_0 M_k \leq (2 N_0)^2 M_{k-1} \leq \cdots \leq (2 N_0)^{k+1} M_0 \; ;$$

in these inequalities

$$M_0 \leq N_0 A , \quad A = \operatorname{Max} | f - C | .$$

From all the preceding it follows that series (34) converges uniformly if

$$|h| < \frac{1}{2N_0} = \lambda_0 \tag{36}$$

and that this series provides the solution of the problem for values of h which in absolute value satisfy the last inequality.

We shall now find an upper bound for $|V|$. Let q be an upper bound for $|f|$. We put

$$w_1 = V - \frac{g}{h}, \qquad w_2 = V + \frac{g}{h}.$$

The functions (36) satisfy the conditions

$$\left.\begin{aligned} \Delta w_1 &= 0 \\ \Delta w_2 &= 0 \end{aligned}\right\} \text{ in } (D_i), \qquad \left.\begin{aligned} \frac{dw_1}{dn} + hw_1 + g - f &= 0 \\ \frac{dw_2}{dn} + hw_2 - g - f &= 0 \end{aligned}\right\}. \text{ on } (S).$$

Let M_1 be the point of (D_i) at which w_1 assumes its maximum and M_2 the point at which w_2 assumes its minimum; since w_1 and w_2 are harmonic functions, the points M_1 and M_2 lie on (S). $\frac{dw_1}{dn}$ is positive at M_1; from the condition $g - f \geqq 0$ at this point

$$w_1 < 0, \qquad V < \frac{g}{h}.$$

$\frac{dw_2}{dn}$ is negative at M_2; from the condition $g + f \geqq 0$ at M_2

$$w_2 > 0, \qquad V > -\frac{g}{h}$$

The function V assumes its maximum at M_1 and its minimum at M_2, since $\frac{g}{h}$ is a constant. Hence, in the entire region (D_i)

$$-\frac{g}{h} < V < \frac{g}{h}, \qquad |V| < \frac{g}{h}.$$

We shall assume that h_0 is a number satisfying the inequality $0 < h_0 < \lambda_0$; let

$$h = h_0 + \eta.$$

In this case equations (28′) and (29′) take the following form:

$$\Delta V = 0 \text{ in } (D_i), \qquad \frac{dV_i}{dn} + \eta V = f - h_0 V \text{ on } (S).$$

We put

$$V = w_0 + \eta w_1 + \eta^2 w_2 + \cdots. \tag{37}$$

For $w_0, w_1, w_2, \ldots$ we then obtain by comparing coefficients of powers of η the conditions

$$\left.\begin{array}{l}\Delta w_0 = 0 \\ \Delta w_1 = 0 \\ \Delta w_2 = 0 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\end{array}\right\} \text{ in } (D_i), \qquad \left.\begin{array}{l}\dfrac{dw_0}{dn} + h_0 w_0 = f \\ \dfrac{dw_1}{dn} + h_0 w_1 = -w_0 \\ \dfrac{dw_2}{dn} + h_0 w_2 = -w_1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\end{array}\right\} \text{ on } (S).$$

But now since the quantity λ_0 of the series (34) is independent of f and h_0 is less than λ_0, we are able to determine all the functions $w_0, w_1, w_2, \ldots$; in particular, one obtains w_0 by adding the expression $\frac{1}{h_0 S}\int_{(S)} f d\sigma$ to the sum of series (34) for $h = h_0$.

Let g_k be the upper bound of $| w_k |$ in (D_i) $(k = 0, 1, 2, \ldots)$; from the remark above

$$g_k < \frac{g_{k-1}}{h_0},$$

whence it follows that

$$| w_k | \leq g_k < \frac{g}{h_0^{k+1}} \qquad (k = 0, 1, 2, \ldots).$$

The radius of convergence of series (37) is therefore no less than h_0; this makes it possible to compute the solution for positive values of h less than $2\lambda_0$. Continuing in this manner, one obtains the solution for every positive value of h.

§8. GREEN'S Function in the Problem of Stationary Temperature

Just as in §1, let R be the distance of the point $M(x,y,z)$ in the interior of (D) from the point $M'(\xi,\eta,\zeta)$; it is desired to find the function of the point M' harmonic in the interior of (D) which satisfies the following condition on the surface (S):

$$\frac{d\Gamma}{dn} + h\Gamma + \frac{d\frac{1}{R}}{dn} + h\frac{1}{R} = 0. \tag{38}$$

From the preceding considerations it follows that one can determine the function Γ by setting f equal to

$$-\frac{d\frac{1}{R}}{dn} - h\frac{1}{R}.$$

We further put

$$G(M, M') = \frac{1}{R} + \Gamma.$$

The function G satisfies the condition

$$\frac{dG}{dn} + hG = \frac{d\frac{1}{R}}{dn} + \frac{d\Gamma}{dn} + h\frac{1}{R} + h\Gamma = 0 \quad \text{on } (S). \tag{39}$$

Comparing G with the function (3) of §1 and repeating the considerations there, we can prove that $G(M,M')$ is a symmetric function. It suffices to verify that the functions

$$G_1 = G(M_1, M'), \quad G_2 = G(M_2, M')$$

satisfy the condition

$$\int_{(S)} \left(G_1 \frac{dG_2}{dn} - G_2 \frac{dG_1}{dn}\right) d\sigma = 0 .$$

However, the last equation is a direct consequence of condition (39).

Just as for the NEUMANN function, one can prove that

$$\int_{(D)} [G(M, M')]^2 d\tau < A . \tag{40}$$

Indeed, this follows from the fact that one can likewise derive the inequality

$$\int_{(S)} |\varrho| d\sigma \leq C \int_{(S)} |f| d\sigma$$

for the solution of equation (31) in §7 for $\lambda = -1$; to obtain inequality (40) from this one need only repeat the arguments presented for the NEUMANN function.

We shall now assume that the potential of the simple layer V satisfies condition (29') of §7 on (S). On (S) then from (39)

$$V\frac{dG}{dn} - G\frac{dV_i}{dn} = V(-hG) - G(-hV + f)$$

$$= -Gf = -G(M, M')f(M') .$$

To the functions V and $G(M,M')$ we now apply GREEN's identity for the subregion of (D_i) which lies outside a sphere (ω) of radius ϱ about M which is entirely contained in (D_i). Since ΔV and ΔG vanish in $(D - \omega)$, we obtain the equation

$$\int_{(\sigma)} \left(V\frac{dG}{dn} - G\frac{dV}{dn}\right) d\sigma = -\int_{(S)} G(M, M')f(M') d\sigma ,$$

where (σ) is the surface of the sphere (ω). and the outer normal is chosen on (σ).

Since the right-hand side of the last equation is independent of ϱ, this also holds for the left-hand side. From its form, however, the left-hand side depends on ϱ; just as in the proof of the symmetry property of the GREEN's function in §1, one can show that the left-hand side tends to $-4\pi V(M)$ as $\varrho \to 0$. Hence, it is equal to $-4\pi V(M)$, and we obtain:

$$V(M) = \frac{1}{4\pi} \int_{(S)} G_i(M, M') f(M') d\sigma .$$

§9. GREEN's Function and POISSON's Equation

Let the function $u(x,y,z)$ have derivatives in (D_i) of first and second order which are continuous up to the boundary (S). Such a function we shall call *regular* in (D_i). In addition, let the function $u(x,y,z)$ satisfy one of the following conditions on the boundary (S):

$$u = 0, \tag{α}$$

$$\frac{d u_i}{d n} + h u = 0, \tag{β}$$

$$\frac{d u_i}{d n} = c \qquad (c = \text{const.}). \tag{γ}$$

Let $G(M,M')$ be the GREEN's function for (D_i) when condition (α) is applicable, the NEUMANN function if condition (γ) is involved, and the GREEN's function for the problem of stationary temperature if we are speaking of condition (β). Let M be a fixed point in (D_i), and let (ω) be a sphere about M with radius ϱ so small that (ω) is entirely contained in (D_i); the surface of this sphere is denoted by (σ). To the region $(D_i - \omega)$ and the functions $G(M,M')$ and $u(M')$ we now apply GREEN's formula:

$$\int\limits_{(D_i-\omega)} (u \Delta G - G \Delta u)\, d\tau = \int\limits_{(S)} \left(u \frac{dG}{dn} - G \frac{du}{dn}\right) d\sigma - \int\limits_{(\sigma)} \left(u \frac{dG}{dn} - G\frac{du}{dn}\right) d\sigma .$$

Here $\Delta G = 0$, and the integral over (S)—except in case (γ)—vanishes because of the conditions which the functions G and $u(M')$ satisfy on (S). Hence,

$$-\int\limits_{(\sigma)} \left(u \frac{dG}{dn} - G \frac{du}{dn}\right) d\sigma = -\int\limits_{(D_i-\omega)} G(M, M') \Delta u\, d\tau + \left(\frac{4\pi}{S} \int\limits_{(S)} u\, d\sigma\right);$$

the last summand here appears only in case (γ).

As in the proof of the symmetry property of GREEN's function, one can easily show that the left-hand side of the last equation has the limit $4\pi u(M)$ as $\varrho \to 0$. The first integral on the right-hand side has as limit the integral over the entire region (D_i) which we shall denote simply by (D). We thus obtain the equation

$$u(M) = -\frac{1}{4\pi} \int\limits_{(D)} G(M, M') \Delta u\, d\tau + \left(\frac{1}{S} \int\limits_{(S)} u\, d\sigma\right), \tag{41}$$

where the second summand is only present in case (γ). Hence, any function $u(M)$ regular in (D) which satisfies one of the conditions (α), (β), (γ) can be represented by formula (41). If there exists a regular solution of the POISSON equation

$$\Delta u = -4\pi\varphi, \tag{42}$$

which satisfies one of the conditions (α), (β), (γ), then this solution is given by the formula

$$u(M) = \int_{(D)} G(M, M') \varphi(M') \, d\tau, \tag{43}$$

where it has been assumed that in case (γ) the solution satisfies the additional condition

$$\int_{(S)} u \, d\sigma = 0 \, . \tag{44}$$

Moreover, for the case (γ) it follows from

$$\int_{(D)} \Delta u \, d\tau = \int_{(S)} \frac{du}{dn} \, d\sigma$$

that

$$c = -\frac{4\pi}{S} \int_{(D)} \varphi(M') \, d\tau \, . \tag{45}$$

From the fact that every regular solution of equation (42) with conditions (α), (β), (γ) can be represented by formula (43) follows the uniqueness of a regular solution of equation (42) with conditions (α), (β), (γ) and the additional conditions (44) and (45) in case (γ). We shall now study the integral (43) under different hypotheses on the function $\varphi(M')$.

Theorem 1. *If φ is bounded and integrable in (D), then the integral* (43) *is a function with H-continuous first derivatives in (D) which on (S) satisfies that condition corresponding to the function $G(M,M')$ chosen; in case (γ) equations* (44) *and* (45) *also hold.*

Proof. For the proof we introduce the Newtonian potential $P[\varphi]$ with density φ. Since φ is bounded, $P[\varphi]$ has H-continuous first derivatives everywhere.

We first establish the validity of the following formula:

$$\int_{(D)} G(M, M') \varphi(M') \, d\tau_{M'} = P(M) + \frac{1}{4\pi} \int_{(S)} \left[P_{(M')} \frac{dG}{dn} - \frac{dP}{dn} G \right] d\sigma \, . \tag{46}$$

For $\Gamma(M,M') = \Gamma(0,1)$:

$$\Gamma(0, 1) = \frac{1}{4\pi} \int_{(S)} \left[\frac{d\Gamma(0, 2)}{dn_2} \frac{1}{r_{12}} - \Gamma(0, 2) \frac{d \frac{1}{r_{12}}}{dn_2} \right] d\sigma_2 \, .$$

We multiply the last identity by $\varphi(1)$, integrate over (D), and interchange the order of integration. Noting then that

$$\int_{(D)} \frac{\varphi(1)}{r_{12}} \, d\tau_1 = P(2) \, , \qquad \int_{(D)} \varphi(1) \frac{d \frac{1}{r_{12}}}{dn_2} \, d\tau_1 = \frac{dP(2)}{dn_2} \, , \tag{47}$$

we find after multiplication by -1:

$$-\int_{(D)} \Gamma(0,1)\,\varphi(1)\,d\tau_1 = \frac{1}{4\pi}\int_{(S)} \left[\Gamma(0,2)\frac{dP(2)}{dn_2} - P(2)\frac{d\Gamma(0,2)}{dn_2}\right] d\sigma_2. \tag{48}$$

Since M is an interior point of the region (D), the functions $P(2)$ and $\frac{1}{r_{02}}$ are harmonic in the region (D_e); from equation (44′) of Chapter I we obtain the following relation:

$$0 = \frac{1}{4\pi}\int_{(S)} \left(P(2)\frac{d\frac{1}{r_{02}}}{dn_2} - \frac{1}{r_{02}}\frac{dP(2)}{dn_2}\right) d\sigma_2. \tag{49}$$

In the first of equations (47) we now replace the index 2 by the index 0 and add to this equation the equations (48) and (49). Since

$$G(0,1) = \frac{1}{r_{01}} - \Gamma(0,1),$$

we then obtain:

$$\int_{(D)} G(0,1)\,\varphi(1)\,d\tau_1 = P(0) + \frac{1}{4\pi}\int_{(S)} \left\{P(2)\frac{dG(0,2)}{dn_2} - G(0,2)\frac{dP(2)}{dn_2}\right\} d\sigma_2. \tag{50}$$

Let V be the potential of an arbitrary simple layer. The following identities then hold:

$$0 = V(0) - \frac{1}{4\pi}\int_{(S)} \left\{\frac{1}{r_{02}}\frac{dV(2)}{dn_2} - V(2)\frac{d\frac{1}{r_{02}}}{dn_2}\right\} d\sigma_2,$$

$$0 = \frac{1}{4\pi}\int_{(S)} \left\{\frac{dV(2)}{dn_2}\Gamma(0,2) - V(2)\frac{d\Gamma(0,2)}{dn_2}\right\} d\sigma_2;$$

on adding these give the identity:

$$0 = V(0) + \frac{1}{4\pi}\int_{(S)} \left\{V(2)\frac{dG(0,2)}{dn_2} - \frac{dV(2)}{dn_2}G(0,2)\right\} d\sigma_2. \tag{51}$$

Now adding (50) and (51) and introducing the notation

$$u(0) = P(0) + V(0), \tag{52}$$

we find

$$\int_{(D)} G(0,1)\,\varphi(1)\,d\tau_1 = u(0) + \frac{1}{4\pi}\int_{(S)} \left\{u(2)\frac{dG(0,2)}{dn_2} - \frac{du(2)}{dn_2}G(0,2)\right\} d\sigma_2. \tag{53}$$

It is easy to show that one can choose simple-layer potentials V_α, V_β, and V_γ in such a manner that $u_\alpha = P + V_\alpha$, $u_\beta = P + V_\beta$, and $u_\gamma = P + V_\gamma$

satisfy conditions (α), (β), (γ) respectively on the boundary (S) and in the last case conditions (44) and (45) as well.

Indeed, $P[\varphi] \in H(1,cA,\lambda)$ everywhere and hence

$$P[\varphi] \in H(1, c_1 A, \lambda')$$

on (S) and $\frac{dP}{dn} \in H(0,c_2A,\lambda')$ on (S); A is here an upper bound for $|\varphi|$ and $0 < \lambda < 1$, $\lambda' < \lambda$.

As follows from the proof of Theorem 1 of IV, §18, the harmonic function which is equal to $-P[\varphi]$ on (S) can be represented as a simple-layer potential which we shall denote by V_α. It was proved there that $V_\alpha \in H(1,c_3A,\lambda'')$ ($\lambda'' < \lambda'$); hence, $u_\alpha = P + V_\alpha$ has H-continuous first derivatives in (D). Moreover, on (S) it is clear that $u_\alpha = 0$. If now in formula (53) $u = u_\alpha$ and $G(0,1)$ is the GREEN's function for the DIRICHLET problem, then we find:

$$\int_{(D)} G(0, 1)\, \varphi(1)\, d\tau_1 = u_\alpha(0).$$

This proves our assertion for the integral (43) in case (α).

We now proceed to the case (β). We established that the integral equation (31) in §7 for $\lambda = -1$ has exactly one solution. It is easy to see that the solution ϱ of this equation is H-continuous on (S) if the function f is H-continuous on (S). Putting

$$f = -\left(\frac{dP}{dn} + hP\right),$$

we find that the simple-layer potential V_β which satisfies the condition

$$\frac{dV_\beta}{dn} + hV_\beta = -\left(\frac{dP}{dn} + hP\right)$$

on (S) has H-continuous first derivatives in (D); this therefore also holds for the function $u_\beta = P + V_\beta$. The function u_β moreover satisfies condition (β) on (S). If $G(0,1)$ denotes the GREEN's function for the problem of stationary temperature, then

$$u_\beta \frac{dG}{dn} - \frac{du_\beta}{dn} G = u_\beta(-hG) - (-hu_\beta)G = 0 \quad \text{on} \quad (S),$$

and formula (53) gives:

$$\int_{(D)} G(0, 1)\, \varphi(1)\, d\tau_1 = u_\beta(0).$$

This proves the assertion for integral (43) in case (β).

If in case (β) h is not assumed to be constant, but rather H-continuous and positive on (S), then all our considerations for the case (β) remain valid. If on the other hand we assume only that h is positive and continuous, then we are only able to prove that u_β as the sum of the function P and a simple-layer potential with continuous density satisfies condition (β); the first derivatives of u_β are however no longer H-continuous in (D) in general.

We now come to the case (γ). Multiplying GAUSS' integral

$$-4\pi = \int\limits_{(S)} \frac{d\frac{1}{r_{10}}}{d n_1} d\sigma_1$$

by $\varphi(0)$, integrating over (D), and taking note of the second of formulas (47), we obtain:

$$\int\limits_{(S)} \frac{dP}{dn} d\sigma = -4\pi \int\limits_{(D)} \varphi(0)\, d\tau;$$

from this it follows that

$$\int\limits_{(S)} \left[\frac{dP}{dn} + \frac{4\pi}{S} \int\limits_{(D)} \varphi\, d\tau\right] d\sigma = 0.$$

Hence, there exists a simple-layer potential V_γ which satisfies the conditioı

$$\frac{dV_{\gamma i}}{dn} = -\frac{dP}{dn} - \frac{4\pi}{S} \int\limits_{(D)} \varphi\, d\tau$$

on (S). The function $u_\gamma = P + V_\gamma$ then satisfies the condition

$$\frac{du_\gamma}{dn} = -\frac{4\pi}{S} \int\limits_{(D)} \varphi\, d\tau\,.$$

on (S). Moreover, V_γ is determined up to an additive constant which one may choose in such a manner that the function u_γ satisfies condition (44) on (S). If $G(0,1)$ denotes the NEUMANN function for the inner problem, then

$$u_\gamma \frac{dG}{dn} - \frac{du_\gamma}{dn} G = -\frac{4\pi}{S} u_\gamma + \frac{4\pi}{S} G(0,1) \int\limits_{(D)} \varphi\, d\tau \qquad \text{on} \quad (S);$$

it is clear that the integral over (S) of each of the summands occurring on the right-hand side of the last equation is equal to zero. Then from equation (53)

$$\int\limits_{(D)} G(0,1)\, \varphi(1)\, d\tau_1 = u_\gamma(0)\,.$$

It remains to prove that V_γ and thus also u_γ have H-continuous derivatives of first order in (D). This follows from Theorem 1 of III, §18, since $\dfrac{dP}{dn}$ is H-continuous on (S).

The proof of our assertion concerning the integral (43) has now been completed. If φ is H-continuous in every region (D') which together with its boundary is contained in the interior of (D), then u_α, u_β, and u_γ have continuous second derivatives at every interior point of (D) and satisfy equation (42) and the corresponding boundary condition. From this follows

Theorem 2. *If φ is bounded in (D) and is H-continuous in every region (D') which together with its boundary is contained in (D), then equation* (42) *has exactly one solution which has continuous second derivatives in (D), satisfies one of the conditions* (α), (β), (γ) *on (S), and moreover satisfies equations* (44) *and* (45) *in case* (γ).

Proof. The existence of such a solution follows from the preceding considerations. In each of the problems considered the uniqueness of the solution follows from the fact that the difference v of two solutions of the problem is a harmonic function which satisfies the condition (α), (β), or (γ) and

$$\frac{dv}{dn} = 0, \qquad \int_{(S)} v\, d\sigma = 0$$

respectively. It clearly follows in all cases that $v \equiv 0$. This proves Theorem 2.

Now suppose that $\varphi(M')$ is not bounded in (D), but that $\varphi(M')$ and $[\varphi(M')]^2$ are integrable in (D). We shall say that $\varphi(M')$ is square-integrable. From the definition of the integral of an unbounded function it follows that there exists a sequence $\{(D_k)\}$ $(k = 1, 2, \ldots)$ of regions which together with their boundaries are contained in (D) such that (a) (D_{k+1}) contains (D_k), (b) $\varphi(M')$ is continuous and bounded in (D_k), and

$$\int_{(D_k)} |\varphi(M')|^2 d\tau \to \int_{(D)} \varphi^2 d\tau .$$

converges.

Let the function $\varphi_k(M')$ be so defined that in (D_k) it equals $\varphi(M')$ and is zero outside (D_k). Now for every $\varepsilon > 0$ there exists an n such that for all $k > n$

$$\int_{(D)} |\varphi - \varphi_k|^2 d\tau = \int_{(D-D_k)} |\varphi(M')|^2 d\tau < \varepsilon .$$

We shall show that the integral (43) is a bounded and continuous function in (D) if $\varphi(M')$ is square-integrable. The boundedness of this integral follows from the BUNYAKOVSKII-SCHWARZ inequality:

$$|u(0)|^2 = \left| \int_{(D)} G(0,1)\varphi(1)\, d\tau_1 \right|^2$$
$$\leq \int_{(D)} |G(0,1)|^2 d\tau_1 \cdot \int_{(D)} |\varphi(1)|^2 d\tau_1 < A \cdot \int_{(D)} |\varphi(1)|^2 d\tau_1 = A \cdot \|\varphi\|^2 .$$

Since the functions $\varphi_k(M')$ are bounded, the functions

$$u_k(0) = \int_{(D)} G(0,1)\varphi_k(1)\, d\tau_1 \qquad (k = 1, 2, \ldots)$$

are continuous in (D). We shall show that the sequence of functions $u_k(0)$ converges uniformly to the integral (43). Indeed, making use of the BUNYAKOVSKII-SCHWARZ inequality, we find:

$$|u(0) - u_k(0)| = \left| \int_{(D)} G(0, 1)[\varphi(1) - \varphi_k(1)]\, d\tau_1 \right|$$
$$\leq \sqrt{\int_{(D)} [G(0, 1)]^2 d\tau_1} \sqrt{\int_{(D)} (\varphi - \varphi_k)^2 d\tau}$$
$$< \sqrt{A} \sqrt{\int_{(D-D_k)} |\varphi|^2 d\tau} < \sqrt{A}\,\varepsilon \qquad (k > n);$$

this holds for each point M of (D). Hence $u(M)$ is the limit function of a uniformly convergent sequence of continuous functions, and this implies that $u(M)$ is itself continuous in (D). One can show that in case (α) the function $u(M)$ has limit zero as the point M approaches the boundary (S).

If φ is bounded and continuous in (D), then, as we have seen in II, §15, the Newtonian potential with density φ satisfies the POISSON equation formed with the generalized LAPLACE operator Δ^* in the sense of I. I. PRIVALOV:

$$\Delta^* P = -4\pi\varphi.$$

It is clear that the function $u(M)$ defined by the integral (43) then also satisfies this equation. If φ is square-integrable, then, as was shown in II, §24, the Newtonian potential $P[\varphi]$ satisfies the POISSON equation where the derivative is understood in the sense of S. L. SOBOLEV. In this same sense the function $u(M)$ then also satisfies this equation (42).

Remark. Let us here present the following assertion without proof: If a continuous function w has the property that $\Delta^* w \equiv 0$ (Δ^* is the LAPLACE operator in the sense of I. I. PRIVALOV), then w is a harmonic function (I. I. PRIVALOV, *Subgarmonicheskie funktsii*, Ch. I, §2). If in addition w satisfies boundary condition (α) or (β), then $w \equiv 0$. From this it follows that the equation

$$\Delta^* u = -4\pi\varphi,$$

wherein φ is a continuous and bounded function in (D), in cases (α) and (β) has exactly one continuous solution, and this is given by the formula (43).

We therefore see that if a function $u(M)$ which satisfies boundary condition (α) or (β) and has continuous second derivatives in (D) has the property that its LAPLACE expression is bounded, then it can be represented by formula (41). A similar assertion can be made for condition (γ).

One can also prove that formula (41) is valid for every continuous and bounded function $u(M)$ which satisfies the appropriate boundary condition and has a square-integrable LAPLACE expression.

Supplement. We put

$$\gamma_1(M) = -\frac{1}{D}\int_{(D)} G(M, M')\, d\tau,$$

where $G(M,M')$ denotes the NEUMANN function for the region (D_i).

From what has been said in this section $\gamma_1(M)$ satisfies the equation

$$\Delta \gamma_1 = \frac{4\pi}{D}$$

in the interior of (D), and on (S) it satisfies the conditions

$$\frac{d\gamma_1}{dn} = \frac{4\pi}{S}, \qquad \int\limits_{(S)} \gamma_1 \, d\sigma = 0 .$$

We form the new function

$$G_1(M, M') = G(M, M') + \gamma_1(M) + \gamma_1(M') - \frac{1}{D}\int\limits_{(D)} \gamma_1(M') \, d\tau ,$$

which we shall call a *modified* NEUMANN *function.*

It is clear that the function $G_1(M,M')$ is symmetric, and for fixed M in the interior of (D) it satisfies as a function of the point M' in (D) (except for $M' = M$) the equation

$$\Delta G_1(M, M') = \frac{4\pi}{D};$$

on the boundary (S) it satisfies the condition

$$\frac{dG_1}{dn} = \frac{dG}{dn} + \frac{d\gamma_1}{dn} = -\frac{4\pi}{S} + \frac{4\pi}{S} = 0 ,$$

i.e.,

$$\frac{dG_1}{dn} = 0 . \tag{54}$$

Moreover,

$$\int\limits_{(D)} G_1(M, M') \, d\tau = \int\limits_{(D)} G(M, M') \, d\tau + D\gamma_1(M) + \int\limits_{(D)} \gamma_1(M') \, d\tau - \int\limits_{(D)} \gamma_1(M') \, d\tau = 0 ,$$

i.e.,

$$\int\limits_{(D)} G_1(M, M') \, d\tau = 0 . \tag{55}$$

If $\varphi(M')$ satisfies the conditions of Theorem 2, then

$$u(M) = \int\limits_{(D)} G_1(M, M')\varphi(M') \, d\tau = \int\limits_{(D)} G(M, M')\varphi(M') \, d\tau + \gamma_1(M)\int\limits_{(D)} \varphi(M') \, d\tau + \int\limits_{(D)} \gamma_1(M')\varphi(M') \, d\tau - \frac{1}{D}\int\limits_{(D)} \gamma_1(M') \, d\tau \cdot \int\limits_{(D)} \varphi(M') \, d\tau;$$

hence, in (D) $u(M)$ satisfies the equation

$$\Delta u = -4\pi\varphi + \frac{4\pi}{D}\int\limits_{(D)} \varphi(M') \, d\tau \tag{56}$$

and on the boundary (S) the condition

$$\frac{du}{dn} = -\frac{4\pi}{S}\int\limits_{(D)} \varphi(M') \, d\tau + \frac{4\pi}{S}\int\limits_{(D)} \varphi(M') \, d\tau = 0 ,$$

i.e.,

$$\frac{du}{dn} = 0. \tag{57}$$

If $\varphi(M')$ is bounded and integrable, then $u(M)$ has H-continuous first derivatives in (D) and on (S) satisfies condition (57). Integrating $u(M)$ over (D), interchanging the order of integration, and taking account of the symmetry property of the function $G_1(M,M')$ as well as equation (55), we obtain:

$$\int\limits_{(D)} u(0)\, d\tau = \int\limits_{(D)} \left[\int\limits_{(D)} G_1(0,1)\, \varphi(1)\, d\tau_1 \right] d\tau$$
$$= \int\limits_{(D)} \varphi(1) \left[\int\limits_{(D)} G_1(0,1)\, d\tau \right] d\tau_1 = 0\,,$$

i.e.,

$$\int\limits_{(D)} u(M')\, d\tau = 0\,. \tag{58}$$

If for $\varphi(M')$ the condition

$$\int\limits_{(D)} \varphi(M')\, d\tau = 0\,, \tag{59}$$

holds, then because of (56) $u(M)$ satisfies equation (42) and conditions (57) and (58).

By repeating the considerations at the beginning of this section, one can easily prove that every regular solution of equation (42) with conditions (57) and (58) is given by formula (43) where $G(M,M')$ is to be replaced by $G_1(M,M')$.

In the following sections by condition (γ) we shall mean condition (57). When condition (57) is being considered, we shall then make use of the modified NEUMANN function and denote it simply by $G(M,M')$.

§10. Problems Involving the Equation $\Delta u = Lu + K$

In the following we shall restrict ourselves to inner problems. Let $u(x,y,z)$ be a regular solution of the equation

$$\Delta u = Lu + K\,, \tag{60}$$

where L and K are continuous functions in (D) and L is positive.

Let the function u satisfy one of the following conditions on (S):

$$u = f\,, \tag{α'}$$

$$\frac{du_i}{dn} = -hu + f\,, \tag{β'}$$

$$\frac{du_i}{dn} = f\,; \tag{γ'}$$

here f is a given continuous function on (S).

We shall first of all show that there cannot be two distinct regular functions u_1 and u_2 which satisfy equation (60) and one of the conditions (α'), (β'), (γ'). If there were two such functions, then the difference $v = u_1 - u_2$ would be a regular function satisfying the equation

$$\Delta v = L v \tag{61}$$

and one of the conditions

$$v = 0\,, \tag{α}$$

$$\frac{d v}{d n} + h v = 0\,, \tag{β}$$

$$\frac{d v}{d n} = 0 \tag{γ}$$

on the boundary (S). Then

$$\int\limits_{(D)} L v^2 \, d\tau = \int\limits_{(D)} v \Delta v \, d\tau = \int\limits_{(S)} v \frac{d v}{d n} \, d\sigma - \int\limits_{(D)} \sum \left(\frac{\partial v}{\partial x}\right)^2 d\tau$$

and hence

$$\int\limits_{(D)} \left\{ L v^2 + \sum \left(\frac{\partial v}{\partial x}\right)^2 \right\} d\tau = \int\limits_{(S)} v \frac{d v}{d n} \, d\sigma\,. \tag{62}$$

In the case of conditions (α) and (γ) the right-hand side of (62) is equal to zero. Since a nonnegative function stands under the integral sign on the left-hand side and this integral vanishes, we come to the conclusion that

$$v = \text{const. and } L v^2 = 0\,, \quad \text{i.e. } v \equiv 0 \text{ in } (D)\,.$$

In case (β) since $h > 0$,

$$\int\limits_{(S)} v \frac{d v}{d n} \, d\sigma = - \int\limits_{(S)} h v^2 \, d\sigma \leq 0\,;$$

hence equation (62) can hold here only if $v \equiv 0$.

We have thus proved that two distinct regular solutions of equations (60) which satisfy one of the conditions (α'), (β'), (γ') cannot exist.

Let v_α and v_β be functions harmonic in the interior of (D) which on (S) satisfy the conditions

$$v_\alpha = f\,, \qquad \frac{d v_\beta}{d n} = - h v_\beta + f\,.$$

The differences $V_\alpha = u_\alpha - v_\alpha$ and $V_\beta = u_\beta - v_\beta$ then satisfy the conditions

$$V_\alpha = 0 \tag{α}$$

$$\frac{d V_\beta}{d n} = - h V_\beta \tag{β}$$

on the boundary (S) and the equation

$$\Delta V = \Delta u = Lu + K = L(V + v) + K = LV + (K + Lv) = LV + F,$$

i.e.,
$$\Delta V = LV + F \text{ with } F = K + Lv, \tag{63}$$
in the interior of (D).

If the solution $V(x,y,z)$ is a regular function, then on the basis of the results of the preceding section we may conclude that the equation

$$V(M) = -\frac{1}{4\pi}\int\limits_{(D)} G(M, M')L(M')V(M')\,d\tau + \Phi(M) \tag{64}$$

holds with
$$\Phi(M) = -\frac{1}{4\pi}\int\limits_{(D)} G(M, M')F(M')\,d\tau; \tag{65}$$

V is thus the solution of an integral equation with kernel $G(M,M')L(M')$.

We shall assume that L is bounded and investigate the second iterated kernel of the integral equation obtained; it reads

$$L(M')\int\limits_{(D)} G(M, M_1')L(M_1')\,G(M_1', M')\,d\tau_1 = L(M')\,G_2(M, M').$$

Since the integral of the square of $G(M,M')$ is bounded, $G_2(M,M')$ is a bounded and continuous function of the points M and M', for we found that integral (43) in §9 is a continuous function if $\varphi(M')$ is square-integrable. One may thus apply the well-known FREDHOLM theorem to the integral equation (64). If $\Phi(M)$ is bounded the solution, if it exists, is also bounded.

We shall now assume that L and F are H-continuous in (D). We can then prove that every solution of the integral equation (64) in which $\Phi(M)$ is defined by formula (65) is a regular solution of equation (63) which satisfies boundary condition (α) or (β) according to the choice of the function $G(M,M')$.

Indeed, since V is a bounded solution of the integral equation, the first summand on the right-hand side of (64) is seen to be a function possessing continuous first derivatives and satisfying the corresponding boundary condition. The second summand is a regular solution of the equation

$$\Delta u = F$$

and satisfies the boundary condition.

From equation (64) it follows that the function $V(M)$ has first derivatives and satisfies the boundary condition, for it is the sum of two functions which have these properties. Thus, LV is H-continuous in (D), which means that the first summand on the right-hand side of (64) satisfies the boundary condition and represents a regular function whose LAPLACE expression is equal to LV. Our assertion now follows. It follows further that every square-integrable solution of the homogeneous integral equation

$$V(M) = -\frac{1}{4\pi}\int\limits_{(D)} G(M, M')L(M')V(M')\,d\tau \tag{66}$$

is identically zero.

If we suppose that V is square-integrable and L is bounded, then we can conclude that LV is likewise square integrable; then from a remark of §9 the function V is continuous and bounded. From this it follows as formerly that V is a regular solution of equation (61) with the boundary condition (α) or (β). Such a solution, as we have seen, is however identically zero.

From this now follows that the number $\lambda = -1$ is not an eigenvalue of the following integral equation:

$$V(M) = \frac{\lambda}{4\pi}\int\limits_{(D)} G(M, M')L(M')V(M')\,d\tau + \Phi(M). \tag{67}$$

We now turn to the study of the problem with boundary condition (γ'). Let v_γ be an arbitrary function satisfying boundary condition (γ') and possessing continuous second derivatives in (D). For example,

$$v_\gamma = v - P,$$

is such a function where P is the Newtonian potential with constant density $c = \frac{1}{4\pi D}\int\limits_{(S)} f d\sigma$, and v is a harmonic function which satisfies the condition

$$\frac{d v_i}{d n} = f + \frac{dP}{d n}$$

on (S). The function v exists, since

$$\begin{aligned}\int\limits_{(S)}\left(f + \frac{dP}{d n}\right) d\sigma &= \int\limits_{(S)} f\, d\sigma + \int\limits_{(S)}\frac{dP}{d n}\, d\sigma \\ &= \int\limits_{(S)} f\, d\sigma + \int\limits_{(D)}\Delta P\, d\tau = \int\limits_{(S)} f\, d\sigma - 4\pi c D = 0.\end{aligned}$$

The function

$$V_\gamma = u_\gamma - v_\gamma$$

then satisfies the boundary condition (γ)

$$\frac{dV_\gamma}{d n} = 0$$

as well as equation (63) with

$$F = K + Lv_\gamma - \frac{1}{D}\int\limits_{(S)} f\, d\sigma.$$

We shall assume that V is a regular solution of equation (63) with the boundary condition (γ). Then

$$\int\limits_{(D)}\Delta V\, d\tau = \int\limits_{(S)}\frac{dV}{d n}\, d\sigma = 0$$

and hence

$$\int_{(D)} (LV + F)\, d\tau = 0. \tag{68}$$

From the results of the preceding section we can thus conclude that V satisfies the integral equation (64) where by $G(M,M')$ the modified NEUMANN function is meant. Just as for conditions (α) and (β), we can convince ourselves that $\lambda = -1$ is not an eigenvalue of equation (67) and that integral equation (64) has exactly one solution. We thus conclude that equation (63) for each of conditions (α), (β), (γ) has exactly one solution if we assume that L and F are H-continuous and L is positive in (D).

Under the hypothesis that L is H-continuous in (D) and positive we proved that $\lambda = -1$ is not an eigenvalue of integral equation (67). We shall now show that the requirement of H-continuity is not necessary. Assuming only that L is bounded, positive, and integrable, we can prove that the homogeneous equation (66) has only the zero solution.

Indeed, under these hypotheses on L one can find a sequence of H-continuous functions L_k in (D) such that

$$\lim_{k \to \infty} \int_{(D)} (L - L_k)^2\, d\tau = 0.$$

We consider the sequence of functions

$$V_k = -\frac{1}{4\pi} \int_{(D)} G(M, M')\, L_k(M')\, V(M')\, d\tau,$$

where $V(M')$ is a solution of the homogeneous equation (66).

From equation (66) it follows that V is bounded and has bounded first derivatives; V_k is then regular and satisfies the equation

$$\Delta V_k = L_k V$$

as well as the corresponding boundary condition. Multiplying both sides of the last equation by V_k and integrating, one finds easily that

$$\int_{(D)} \left\{ L_k V V_k + \sum \left(\frac{\partial V_k}{\partial x} \right)^2 \right\} d\tau \leq 0.$$

The functions V_k and the first derivatives of the V_k tend uniformly to V and the corresponding derivatives of V. From this it follows as before that the solution V of the homogeneous equation (66) is identically zero.

It is to be noted that we have now actually proved the assertion that the eigenvalues of the integral equation

$$V(M) = \frac{\lambda}{4\pi} \int_{(D)} G(M, M')\, L(M')\, V(M')\, d\tau$$

are nonnegative. For if $\lambda < 0$, then $\lambda L = -|\lambda| L$, and the homogeneous equation (66) with L replaced by $|\lambda| L$ has only the zero solution as we have just shown.

The integral equation just considered is an equation of SCHMIDT type (the kernel is the product of the symmetric function $G(M,M')$ and the positive function $L(M')$; hence, all the eigenvalues are real, and since they cannot be negative they are all positive. If the function L is H-continuous the eigenfunctions are solutions of the equation

$$\Delta V + \lambda L V = 0$$

and on the boundary (S) satisfy the condition corresponding to the function $G(M,M')$ chosen. It should be mentioned that in case (γ) the functions $V =$ const. satisfy the boundary condition and the equation just mentioned for $\lambda = 0$. We shall therefore call these functions eigenfunctions for the eigenvalue 0 (and condition (γ)).

§11. Lemma

In the integral equation (67) of §10 we shall replace $\frac{L}{4\pi}$ by L. If one determines the solution of the integral equation

$$V(M) = \lambda \int_{(D)} G(M, M') L(M') V(M')\, d\tau + f(M)$$

by the method of successive approximation, one comes to the series

$$V = v_0 + \lambda v_1 + \cdots + \lambda^k v_k + \cdots \tag{69}$$

with

$$v_0 = f,$$

$$v_k(M) = \int_{(D)} G(M, M') L(M')\, v_{k-1}(M')\, d\tau \qquad (k = 1, 2, \ldots).$$

Let l be the radius of convergence of the series (69).

Lemma. *The inequality*

$$\frac{\int_{(D)} L(M) \left(\int_{(D)} G(M, M') L(M') f(M')\, d\tau_{M'} \right)^2 d\tau_M}{\int_{(D)} L(M') [f(M')]^2\, d\tau} \leq \frac{1}{l^2} \tag{70}$$

holds.

Proof. For the proof note first of all that the radius of convergence of the series

$$v_0 + \lambda^2 v_2 + \cdots + \lambda^{2k} v_{2k} + \cdots$$

is not less than l.

Multiplying the last series by Lv_0 and integrating over (D), we obtain the power series

$$\sum_{k=0}^{\infty} \lambda^{2k} \int_{(D)} L v_0 v_{2k} \, d\tau; \tag{71}$$

its radius of convergence l_1 is likewise not less than l.

We introduce the notation

$$I_{k,n} = \int_{(D)} L v_k v_n \, d\tau, \qquad J_k = I_{k,k}.$$

Then

$$\begin{aligned} I_{k,n} &= \int_{(D)} L(M) v_k(M) \left(\int_{(D)} L(M') G(M, M') v_{n-1}(M') \, d\tau_{M'} \right) d\tau_M \\ &= \int_{(D)} L(M') v_{n-1}(M') \left(\int_{(D)} L(M) v_k(M) G(M, M') \, d\tau_M \right) d\tau_{M'} \\ &= \int_{(D)} L(M') v_{n-1}(M') v_{k+1}(M') \, d\tau_{M'} = I_{k+1, n-1}. \end{aligned}$$

From this follows:

$$J_k = I_{k,k} = I_{k-1,k+1} = I_{k-2,k+2} = \cdots = I_{0,2k} = \int_{(D)} L v_0 v_{2k} \, d\tau;$$

so for the series (71) one may also write

$$\sum_{k=0}^{\infty} J_k \lambda^{2k}. \tag{72}$$

We shall show that

$$\frac{J_1}{J_0} \leq \frac{1}{l_1^2} \leq \frac{1}{l^2}; \tag{73}$$

if this has been shown, then the lemma is proved, for the left-hand side of inequality (70) is equal to the ratio of J_1 to J_0.

We assume that f is square-integrable; to assume that f is continuous does not simplify the proof in any way. Let the function L be bounded and positive.

It will first be shown that the kernel $G(M,M')$ is complete, i.e., from the identity

$$\int_{(D)} G(M, M') \varphi(M') \, d\tau \equiv 0,$$

in which $\varphi(M')$ is a square-integrable function, it follows that $\varphi \equiv 0$. If $G(M,M')$ is the modified NEUMANN function, it follows that φ = const.

Let $\psi(M)$ be a regular function in (D) which vanishes outside some arbitrary region (D') which together with its boundary is contained in (D). On (S) $\psi(M)$ then satisfies each of conditions (α), (β), and (γ).

If we multiply the identity above by $\Delta\psi(M)$, integrate over (D), interchange the order of integration, and note that in cases (α) and (β)

$$-\frac{1}{4\pi}\int_{(D)} G(M, M')\,\Delta\psi(M)\,d\tau_M = \psi(M')$$

and in case (γ)

$$-\frac{1}{4\pi}\int_{(D)} G(M, M')\,\Delta\psi(M)\,d\tau_M = \psi(M') - \frac{1}{D}\int_{(D)}\psi(M')\,d\tau\,,$$

then in cases (α) and (β) we obtain the equation

$$\int_{(D)}\varphi(M')\,\psi(M')\,d\tau = 0$$

and in case (γ) the equation

$$\int_{(D)}\psi(M')\left(\varphi(M') - \frac{1}{D}\int_{(D)}\varphi(M)\,d\tau_M\right)d\tau_{M'} = 0.$$

Since the function $\psi(M')$ is arbitrary, it follows that in cases (α) and (β)

$$\varphi(M') \equiv 0;$$

in case (γ) we conclude that

$$\varphi(M') - \frac{1}{D}\int_{(D)}\varphi(M)\,d\tau \equiv 0,$$

i.e.,

$$\varphi(M') = \frac{1}{D}\int_{(D)}\varphi(M)\,d\tau = \text{const.}$$

In the following it will be assumed that f is not identically zero. Then in cases (α) and (β)

$$v_1 = \int_{(D)} G(M, M')\,L(M')f(M')\,d\tau \not\equiv 0\,;$$

continuing this procedure, we obtain successively: $v_k \not\equiv 0$ $(k = 1, 2, \ldots)$. In case (γ) we conclude, if we similarly assume that $Lf \not\equiv$ const., that $v_1 \not\equiv 0$. Since in this case, however, the equation

$$\int_{(D)} v_1\,d\tau = 0$$

also holds, v_1 must assume values of opposite sign. This implies that Lv_1 assumes values of opposite sign and therefore cannot be constant. From this we obtain successively that $Lv_k \not\equiv$ const. $(k = 1, 2, \ldots)$.

These considerations lead to the conclusion:

$$J_k = \int_{(D)} L\,v_k^2\,d\tau > 0 \qquad (k = 0, 1, 2, \ldots)$$

Above we derived the relation

$$J_k = I_{k,k} = I_{k-1, k+1} \quad .$$

Applying the BUNYAKOVSKII-SCHWARZ inequality we find:

$$I_{k-1,k+1}^2 = \Big(\int\limits_{(D)} L v_{k+1} v_{k-1} d\tau\Big)^2 \leq \int\limits_{(D)} L v_{k+1}^2 d\tau \int\limits_{(D)} L v_{k-1}^2 d\tau = J_{k+1} J_{k-1};$$

using the preceding equation, we now obtain the estimate:

$$J_k^2 \leq J_{k+1} J_{k-1}.$$

Since $J_k > 0$, it follows that

$$\frac{J_k}{J_{k-1}} \leq \frac{J_{k+1}}{J_k},$$

i.e., the ratio

$$\frac{J_k}{J_{k-1}}$$

decreases with increasing k and thus as $k \to \infty$ tends to a finite or infinite limit which is equal to the inverse of the radius of convergence l_1 of series (72).

All these ratios are no greater than the limit mentioned above; for $k = 1$ we obtain inequality (73). This completes the proof of the lemma.

§12. Remarks on the Poles of the Solution of the Integral Equation (67)

Let

$$\frac{D(\lambda, M, M')}{D(\lambda)} \tag{74}$$

be the resolvent for the second iterated kernel of integral equation (67) in §10. Since the equation is of SCHMIDT type, all poles of this resolvent are simple. The solution of the integral equation in question has the form

$$V = \frac{D_1(\lambda, M)}{D_1(\lambda)}, \tag{75}$$

where we assume that this fraction is in lowest form. The function $D_1(\lambda)$ is obtained from $D(\lambda)$ after division by a certain entire function; $D_1(\lambda)$ has no zeros of order greater than one.

We previously found that the eigenvalues of the integral equation under consideration are positive. Let

$$\lambda_1, \lambda_2, \ldots, \lambda_k, \ldots, \lambda_n, \ldots \tag{76}$$

be these eigenvalues; suppose that they are ordered in a nondecreasing sequence in which each eigenvalue is repeated as often as there are linearly independent eigenfunctions corresponding to it. Let

$$V_1, V_2, \ldots, V_k, \ldots, V_n, \ldots \tag{77}$$

be the sequence of corresponding eigenfunctions which we assume are orthogonal and normalized, i.e.,

$$\int_{(D)} \bar{L}(0)\, V_k(0)\, V_n(0)\, d\tau = 0, \quad \text{if} \quad k \neq n,$$

$$\int_{(D)} \bar{L}(0)\, V_k^2(0)\, d\tau = 1, \qquad \bar{L} = \frac{1}{4\pi} L.$$

Let it further be assumed that the function Φ satisfies the n conditions

$$\int_{(D)} \bar{L}\,\Phi\, V_k\, d\tau = 0 \qquad (k = 1, 2, \ldots, n)\,. \tag{78}$$

We shall prove that in this case the numbers

$$\lambda_1, \lambda_2, \ldots, \lambda_n,$$

where $\lambda_{n+1} > \lambda_n$ are not poles of the fraction (75).

It is clear that (75) satisfies the equation

$$D_1(\lambda, 0) = \frac{\lambda}{4\pi} \int_{(D_1)} L(1)\, G(1, 0)\, D_1(\lambda, 1)\, d\tau_1 + D_1(\lambda)\,\Phi(0)\,. \tag{79}$$

If λ_k is a zero of $D_1(\lambda)$, then

$$D_1(\lambda_k, 0) = \frac{\lambda_k}{4\pi} \int_{(D_1)} L(1)\, G(1, 0)\, D_1(\lambda_k, 1)\, d\tau_1.$$

From this it follows that the function $D_1(\lambda_k,0)$ is an eigenfunction corresponding to the eigenvalue λ_k; it is a linear combination of the functions (77).

For $k \leqq n$ from (78):

$$\int_{(D)} L(0)\,\Phi(0)\, D_1(\lambda_k, 0)\, d\tau = 0.$$

Multiplying both sides of (79) by $L(0)\,D_1(\lambda_k,0)$ and integrating the product over (D), we obtain:

$$\int_{(D)} L(0)\, D_1(\lambda_k, 0)\, D_1(\lambda, 0)\, d\tau$$

$$= \frac{\lambda}{4\pi} \int_{(D)} L(0)\, D_1(\lambda_k, 0) \left(\int_{(D_1)} L(1)\, G(1, 0)\, D_1(\lambda, 1)\, d\tau_1 \right) d\tau$$

$$= \frac{\lambda}{4\pi} \int_{(D_1)} L(1)\, D_1(\lambda, 1) \left(\int_{(D)} L(0)\, G(1, 0)\, D_1(\lambda_k, 0)\, d\tau \right) d\tau_1$$

$$= \frac{\lambda}{\lambda_k} \int_{(D_1)} L(1)\, D_1(\lambda, 1)\, D_1(\lambda_k, 1)\, d\tau_1\,;$$

from this it follows that

$$\left(1-\frac{\lambda}{\lambda_k}\right)\int\limits_{(D)} L(0)\, D_1(\lambda_k, 0)\, D_1(\lambda, 0)\, d\tau = 0.$$

If $\lambda \neq \lambda_k$ then it follows that

$$\int\limits_{(D)} L(0)\, D_1(\lambda_k, 0)\, D_1(\lambda, 0)\, d\tau = 0.$$

But this integral is an entire function of λ; since for $\lambda \neq \lambda_k$ it is equal to zero, it must also vanish for $\lambda = \lambda_k$. Hence,

$$\int\limits_{(D)} L(0)\, [D_1(\lambda_k, 0)]^2 d\tau = 0,$$

from which $D_1(\lambda_k,0) = 0$ follows. This is a contradiction, since the fraction (75) is irreducible. Hence, the numbers $\lambda_k (k = 1, 2, \ldots, n)$ cannot be poles of the fraction (75).

§13. Closedness of the Sequence of Eigenfunctions in a Special Function Space

We shall say that the sequence (77) of §12 is *closed* in a function space whose elements possess certain given properties if for every function $f(0)$ of this space the equation

$$\frac{1}{4\pi}\int\limits_{(D)} L(0)\, [f(0)]^2 d\tau = \sum_{k=1}^{\infty}\left[\frac{1}{4\pi}\int\limits_{(D)} L(0)\, V_k(0) f(0)\, d\tau\right]^2 = \sum_{k=1}^{\infty} a_k^2 \qquad (80)$$

holds in which

$$a_k = \frac{1}{4\pi}\int\limits_{(D)} L(0)\, V_k(0) f(0)\, d\tau$$

From the relation

$$0 \leq \frac{1}{4\pi}\int\limits_{(D)} L(0)\left(f - \sum_{k=1}^{n} a_k V_k\right)^2 d\tau = \frac{1}{4\pi}\int\limits_{(D)} L(0)\, [f(0)]^2 d\tau - \sum_{k=1}^{n} a_k^2$$

the difference

$$\frac{1}{4\pi}\int\limits_{(D)} L(0)\, [f(0)]^2 d\tau - \sum_{k=1}^{n} a_k^2$$

is nonnegative for every square-integrable function f. For

$$\sum_{k=1}^{n} a_k^2 \leq \frac{1}{4\pi}\int\limits_{(D)} L(0)\, [f(0)]^2 d\tau$$

and hence

$$\sum_{k=1}^{\infty} a_k^2 \leq \frac{1}{4\pi}\int\limits_{(D)} L(0)[f(0)]^2\, d\tau.$$

The last relation is called BESSEL's *inequality*.

Let $h(0)$ be some square-integrable function. We put

$$f(0) = \frac{1}{4\pi}\int\limits_{(D)} L(1)\, G(0,\, 1)\, h(1)\, d\tau_1 \tag{81}$$

and wish to prove that in the space of functions $f(0)$ having the form (81) the sequence (77) in §12 is closed. If we set

$$\left.\begin{aligned} h(0) &= \sum_{k=1}^{n} a_k V_k(0) + \varrho_n(0), \\ a_k &= \frac{1}{4\pi}\int\limits_{(D)} L(0)\, h(0)\, V_k(0)\, d\tau, \end{aligned}\right| \tag{82}$$

then we obtain

$$\begin{aligned} f(0) &= \frac{1}{4\pi}\int\limits_{(D_1)} L(1)\, G(1,\, 0)\left(\sum_{k=1}^{n} a_k V_k(1)\right) d\tau_1 \\ &\quad + \frac{1}{4\pi}\int\limits_{(D_1)} L(1)\, G(1,\, 0)\, \varrho_n(1)\, d\tau_1 \qquad (83) \\ &= \sum_{k=1}^{n} \frac{a_k}{\lambda_k} V_k(0) + R_n(0) = \sum_{k=1}^{n} b_k V_k(0) + R_n(0). \end{aligned}$$

Here

$$R_n(0) = \frac{1}{4\pi}\int\limits_{(D_1)} L(1)\, G(1,\, 0)\, \varrho_n(1)\, d\tau_1,$$

$$b_k = \frac{1}{4\pi}\int\limits_{(D)} L(0) f(0)\, V_k(0)\, d\tau = \frac{a_k}{\lambda_k}.$$

We multiply both sides of the first equation in (82) by $L(0)V_m(0)$ ($m = 1, 2, \ldots, n$) and integrate the product so obtained over (D). Then since the sequence $\{V_k\}$ is orthonormal:

$$\begin{aligned} \int\limits_{(D)} L(0)\, h(0) V_m(0)\, d\tau &= \sum_{k=1}^{n} a_k \int\limits_{(D)} L(0)\, V_k(0) V_m(0)\, d\tau + \int\limits_{(D)} L(0)\, \varrho_n(0)\, V_m(0)\, d\tau \\ &= 4\pi a_m + \int\limits_{(D)} L(0)\, \varrho_n(0)\, V_m(0)\, d\tau; \end{aligned}$$

from this it follows that

$$\int\limits_{(D)} L(0)\, \varrho_n(0) V_m(0)\, d\tau = 0 \qquad (m = 1, 2, \ldots, n). \tag{84}$$

As in §11, we now form the series

$$V = v_0 + \lambda v_1 + \cdots + \lambda^k v_k + \cdots, \tag{85}$$

which corresponds to the equation

$$V(0) = \frac{\lambda}{4\pi}\int_{(D_1)} L(1)\,G(1,0)\,V(1)\,d\tau_1 + \varrho_n(0)\,.$$

From §12 the radius of convergence of this series is greater than $|\lambda_n|$; indeed, $\lambda_1, \lambda_2, \ldots, \lambda_n$ are not poles of $V(0)$ from equations (84).

From the lemma in §11

$$\frac{\int_{(D)} L(0)\left(\int_{(D_1)} L(1)\,G(1,0)\,\varrho_n(1)\,d\tau_1\right)^2 d\tau}{(4\pi)^2\int_{(D)} L(0)\,[\varrho_n(0)]^2\,d\tau} = \frac{\int_{(D)} L(0)\,[R_n(0)]^2\,d\tau}{\int_{(D)} L(0)\,[\varrho_n(0)]^2\,d\tau} < \frac{1}{\lambda_n^2}. \tag{86}$$

Two cases are now to be distinguished:

1. If the number N of the eigenvalues is finite, then the series (85) converges for every arbitrary λ if $n = N$. The radius of convergence of the series is then equal to $+\infty$; from inequality (86) it follows that

$$\int_{(D)} L(0)\,[R_N(0)]^2\,d\tau = 0\,, \quad R_N = 0$$

and

$$f(0) = \frac{1}{4\pi}\int_{(D)} L(1)\,G(1,0)\,h(1)\,d\tau = b_1 V_1(0) + b_2 V_2(0) + \cdots + b_N V_N(0)\,.$$

2. If the number of eigenvalues is infinite, then since the function V is meromorphic $\lambda_n \to \infty$ as $n \to \infty$. From this there follows:

$$\lim_{n\to\infty}\int_{(D)} L(0)\,[R_n(0)]^2\,d\tau = 0\,.^{2})$$

From the relation

$$\int_{(D)} L(0)[R_n(0)]^2\,d\tau = \int_{(D)} L(0)\,[f(0)]^2\,d\tau - 4\pi\sum_{k=1}^{n} b_k^2$$

we may write the equation obtained in the form

$$\frac{1}{4\pi}\int_{(D)} L(0)\,[f(0)]^2\,d\tau = \sum_{k=1}^{\infty} b_k^2 = \sum_{k=1}^{\infty}\left(\frac{1}{4\pi}\int_{(D)} L(0)\,V_k(0)\,f(0)\,d\tau\right)^2.$$

This proves that the sequence of eigenfunctions of (67) is closed in the class of functions which can be represented by the integral (81).

Let now f be a regular function in (D) which on (S) satisfies the boundary condition appropriate to the function $G(M,M')$ chosen; in case (γ) it shall be moreover assumed that the integral of f over (D) vanishes. As we have seen at the beginning of §9, the equation

$$f(0) = -\frac{1}{4\pi}\int_{(D)} G(0,1)\,\Delta f(1)\,d\tau_1$$

holds for f; hence,

$$f(0) = \frac{1}{4\pi}\int\limits_{(D)} L(1)\, G(0, 1)\, h(1)\, d\tau_1 \quad \text{with} \quad h(1) = -\frac{\Delta f(1)}{L(1)}.$$

Due to the regularity of f, $\Delta f(1)$ is bounded and continuous in (D). If we assume that L has a positive lower bound α $(L > \alpha)$, then $h(M')$ is likewise bounded. Thus $f(M')$ can be represented by the integral (81), and we arrive at the following conclusion: For every regular function in (D) which on (S) satisfies the corresponding boundary condition the closedness condition (80) is satisfied.

§14. The Closedness of the Sequence of Eigenfunctions

We prove that the sequence of eigenfunctions is closed only for the space consisting of functions which are continuous in (D) up to the boundary. Let $f(0)$ be such a function. For every $\varepsilon > 0$ there exists a polynomial $P(0)$ (polynomial in x,y,z) such that in the region (D) $|f - P| < \varepsilon$ and $|P| < M$ where M is an upper bound for $|f|$. Further let $\omega(0)$ be a regular function in (D) with the following properties: (a) $\omega(0)$ is equal to one in a certain subregion (D') of (D) which together with its boundary is entirely contained in (D); (b) $\omega(0)$ is equal to zero in a certain region (D'') which together with its boundary is likewise contained in (D) and which contains (D') in its interior; (c) the inequality $|\omega(0)| \leqq 1$ holds in (D). How one goes about actually constructing such a function for arbitrary regions (D') and (D'') contained in (D) will not be treated here. We shall also assume that the volume of the region $(D - D')$, i.e., the complementary region of (D') with respect to (D), is less than ε. The function $F = P(0)\omega(0)$ is then regular in (D); in (D') it coincides with P and is equal to zero outside (D''). Hence, the function F satisfies each of the boundary conditions (α), (β), (γ). Moreover, $|F| \leqq M$. Let A be an upper bound for L; we then obtain:

$$\int\limits_{(D)} L(0)\,[f(0) - F(0)]^2\, d\tau = \int\limits_{(D')} L(0)\,[f(0) - F(0)]^2\, d\tau$$
$$+ \int\limits_{(D-D')} L(0)\,[f(0) - F(0)]^2\, d\tau < AD\,\varepsilon^2 + A \cdot 4\,M^2\varepsilon < a\,\varepsilon;$$

hence,

$$\int\limits_{(D)} L(0)\,[f(0) - F(0)]^2\, d\tau < a\,\varepsilon. \tag{87}$$

In case (γ) let it be moreover assumed that the integrals over (D) of the functions f and F vanish.

The function F can be represented as follows:

$$F = -\frac{1}{4\pi}\int\limits_{(D_1)} G(1, 0)\,\Delta F\,d\tau_1 \;;$$

thus,

$$\left.\begin{aligned} &\frac{1}{4\pi}\int\limits_{(D)} L(0)\,[F(0)]^2\,d\tau - \sum_{k=1}^{\infty} a_k^2 = 0, \\ &a_k = \frac{1}{4\pi}\int\limits_{(D)} L F V_k\,d\tau . \end{aligned}\right\} \tag{88}$$

We can write the first equation in (88) in the following manner:

$$\frac{1}{4\pi}\int\limits_{(D)} LF^2\,d\tau - \sum_{k=1}^{\infty} a_k^2 = \frac{1}{4\pi}\int\limits_{(D)} Lf^2\,d\tau + \frac{1}{4\pi}\int\limits_{(D)} L(f-F)^2\,d\tau$$
$$+ \frac{2}{4\pi}\int\limits_{(D)} Lf(F-f)\,d\tau - 2\sum_{k=1}^{\infty} b_k c_k - \sum_{k=1}^{\infty} c_k^2 - \sum_{k=1}^{\infty} b_k^2 = 0$$

with

$$b_k = \frac{1}{4\pi}\int\limits_{(D)} Lf V_k\,d\tau ,$$

$$c_k = a_k - b_k = \frac{1}{4\pi}\int\limits_{(D)} L(F-f)V_k\,d\tau .$$

From the last identity we obtain:

$$\frac{1}{4\pi}\int\limits_{(D)} Lf^2\,d\tau - \sum_{k=1}^{\infty} b_k^2$$
$$= -\frac{1}{4\pi}\int\limits_{(D)} L(f-F)^2\,d\tau - \frac{2}{4\pi}\int\limits_{(D)} Lf(F-f)\,d\tau + 2\sum_{k=1}^{\infty} b_k c_k + \sum_{k=1}^{\infty} c_k^2 .$$

From this there follows the inequality:

$$0 \le \frac{1}{4\pi}\int\limits_{(D)} Lf^2\,d\tau - \sum_{k=1}^{\infty} b_k^2$$
$$\le \frac{1}{4\pi}\int\limits_{(D)} L(f-F)^2\,d\tau + \frac{2}{4\pi}\int\limits_{(D)} L|f|\,|F-f|\,d\tau + 2\sum_{k=1}^{\infty} |b_k|\,|c_k| + \sum_{k=1}^{\infty} c_k^2 .$$

From (87) the following inequalities are obtained:

$$\int\limits_{(D)} L(f-F)^2\,d\tau < a\varepsilon ,$$

$$\int\limits_{(D)} L|f|\,|F-f|\,d\tau \le \sqrt{\int\limits_{(D)} Lf^2\,d\tau \int\limits_{(D)} L(F-f)^2\,d\tau} < \sqrt{A M^2 D a \varepsilon} ,$$

$$\sum_{k=1}^{\infty} c_k^2 \leq \frac{1}{4\pi} \int_{(D)} L(f-F)^2 d\tau < \frac{1}{4\pi} a \varepsilon ,$$

$$\sum_{k=1}^{\infty} |b_k| |c_k| \leq \sqrt{\sum_{k=1}^{\infty} b_k^2 \sum_{k=1}^{\infty} c_k^2}$$
$$\leq \sqrt{\frac{1}{4\pi} \int_{(D)} L f^2 d\tau \cdot \frac{1}{4\pi} \int_{(D)} L(f-F)^2 d\tau} < \frac{1}{4\pi} \sqrt{A} M^2 D a \varepsilon .$$

Hence,

$$0 \leq \frac{1}{4\pi} \int_{(D)} L f^2 d\tau - \sum_{k=1}^{\infty} b_k^2 < b \sqrt{\varepsilon}$$

(b is some number independent of ε). Since ε is arbitrary, from the last inequality we obtain the equation:

$$\frac{1}{4\pi} \int_{(D)} L f^2 d\tau = \sum_{k=1}^{\infty} b_k^2 ;$$

this is what was required to prove.

We have proved that the sequence of eigenfunctions is closed in the class of functions continuous up to the boundary. We recall the following theorem due to STEKLOV: *If a sequence of orthonormal functions is closed in the class of continuous functions, then this sequence is also closed in the class of square-summable functions, i.e., closed in L_2.*[1] Thus from the STEKLOV theorem and the results established in this section the sequence of eigenfunctions is closed in the class L_2.

It is noted that the problem of closedness of orthonormal sequences has its most complete solution in L_2. It is just in this space that to every sequence of numbers which is square-summable there exists a uniquely determined function of L_2 which has these numbers as FOURIER coefficients with respect to some closed, orthonormal sequence of functions. There is therefore a one-to-one correspondence between all sequences of numbers with the property mentioned and all functions of the class L_2.

[1] Cf. I. P. NATANSON, *Theory of functions of a real variable* (Frederick Ungar, New York, 1961), Vol. I, Ch. VII, §3, Theorem 2 (Trans.).

§15. On Expansion in Terms of Eigenfunctions

Let f and g be two square-integrable functions. Their sum is then likewise square-integrable; hence,

$$\int\limits_{(D)} \overline{L}(f+g)^2 d\tau = \sum_{k=1}^{\infty}(a_k+b_k)^2$$

or

$$\int\limits_{(D)} \overline{L} f^2 d\tau + \int\limits_{(D)} \overline{L} g^2 d\tau + 2\int\limits_{(D)} \overline{L} f g\, d\tau = \sum_{k=1}^{\infty} a_k^2 + \sum_{k=1}^{\infty} b_k^2 + 2\sum_{k=1}^{\infty} a_k b_k$$

with

$$\overline{L} = \frac{L}{4\pi}, \qquad a_k = \int\limits_{(D)} \overline{L} f V_k d\tau, \qquad b_k = \int\limits_{(D)} \overline{L} g V_k d\tau .$$

Since

$$\int\limits_{(D)} \overline{L} f^2 d\tau = \sum_{k=1}^{\infty} a_k^2, \qquad \int\limits_{(D)} \overline{L} g^2 d\tau = \sum_{k=1}^{\infty} b_k^2$$

we find from the preceding equation:

$$\int\limits_{(D)} \overline{L} f g\, d\tau = \sum_{k=1}^{\infty} a_k b_k . \tag{89}$$

Let (ω) be a region in the interior of (D). We put $g(M)$ equal to $\frac{1}{\omega}$ in (ω) and equal to zero outside (ω). It is clear that $g(M)$ is square-integrable. Applying equation (89), we obtain:

$$\frac{1}{\omega}\int\limits_{(\omega)} \overline{L} f d\tau = \sum_{k=1}^{\infty} a_k \left(\frac{1}{\omega}\int\limits_{(\omega)} \overline{L} V_k d\tau\right).$$

We shall call the expression

$$\frac{1}{\omega}\int\limits_{(\omega)} \overline{L} f d\tau$$

the mean value of the function f (with weight $\overline{L}$) over the region (ω).

The last equation means then that the mean value of the function f over an arbitrary region (ω) can be expanded in a series of mean values of the eigenfunctions. The expression

$$\frac{1}{\omega}\int\limits_{(\omega)} f d\tau$$

will be called the mean value of the function f over the region (ω) in the strict sense.

If we assume that L is bounded below by a positive number α and that the inequality

$$\bar{L} \geq \alpha > 0$$

holds, and if we put $g = \frac{1}{\bar{L}}$ in (ω) and equal to zero outside (ω), then in analogy to the preceding we obtain:

$$\frac{1}{\omega}\int\limits_{(\omega)} f\,d\tau = \sum_{k=1}^{\infty} a_k \left(\frac{1}{\omega}\int\limits_{(\omega)} V_k\,d\tau\right).$$

This means that the mean value of the function f over the region (ω) in the strict sense can likewise be expanded in a series of such mean values of the eigenfunctions.

The two results just established constitute a generalization of the following theorem due to STEKLOV to the case of square-integrable functions.

Theorem (STEKLOV). *If the function $f(M)$ is continuous in (D), then its mean value can be expanded in a series of mean values of the eigenfunctions.*

We now proceed to a theorem of HILBERT and SCHMIDT.

Theorem (HILBERT-SCHMIDT). *The function*

$$f(M) = \frac{1}{4\pi}\int\limits_{(D)} L(M')\,G(M, M')\,h(M')\,d\tau, \tag{90}$$

where $h(M)$ is a square-integrable function, can be expanded in an absolutely and uniformly convergent series of eigenfunctions.

Proof. First of all let us recall that in §13 we proved that the system of eigenfunctions was closed in the class of functions which can be represented by the integral (90). From this it follows that to every $\varepsilon > 0$ there exists an N such that for $n \geqq N$ the inequality

$$\int\limits_{(D)} \bar{L}\left(f - \sum_{k=1}^{n} a_k V_k\right)^2 d\tau < \varepsilon \tag{91}$$

holds with

$$a_k = \int\limits_{(D)} \bar{L} f V_k\,d\tau = \frac{1}{\lambda_k}\,b_k = \frac{1}{\lambda_k}\int\limits_{(D)} \bar{L} h V_k\,d\tau.$$

For each fixed M the function $G(M,M')$ as a function of the point M' is square-integrable. Moreover,

$$c_k(M) = \int\limits_{(D)} \bar{L}(M')\,G(M, M')\,V_k(M')\,d\tau = \frac{V_k(M)}{\lambda_k}$$

and hence from BESSEL's inequality

$$\sum_{k=1}^{\infty}[c_k(M)]^2 = \sum_{k=1}^{\infty}\frac{[V_k(M)]^2}{\lambda_k^2} \leq \int_{(D)} \overline{L}[G(M, M')]^2 d\tau < A\,.$$

Also from BESSEL's inequality the series

$$\sum_{k=1}^{\infty} b_k^2$$

is convergent; thus, to every given $\varepsilon > 0$ there exists an N_1 such that for $n \geqq N_1$

$$\sum_{k=n+1}^{\infty} b_k^2 < \varepsilon\,.$$

We shall now show that the series $\sum_{k=1}^{\infty} a_k V_k$ is absolutely and uniformly convergent. Indeed, if $n \geqq N_1$ the following estimate is valid independent of the choice of p:

$$\sum_{k=n+1}^{n+p} |a_k V_k| = \sum_{k=n+1}^{n+p} |b_k| \cdot \left|\frac{V_k}{\lambda_k}\right|$$

$$\leq \sqrt{\sum_{k=n+1}^{n+p} b_k^2 \sum_{k=n+1}^{n+p} \frac{V_k^2}{\lambda_k^2}} \leq \sqrt{\sum_{k=1}^{\infty}\frac{V_k^2}{\lambda_k^2}}\,\sqrt{\sum_{k=n+1}^{n+p} b_k^2} < \sqrt{A}\,\varepsilon\,.$$

This then proves the absolute and uniform convergence of the series

$$\sum_{k=1}^{\infty} a_k V_k = \varphi$$

the sum of this series, which we have denoted by φ, is a continuous function. It remains to verify that $f = \varphi$. This is indeed the case, since we are able to prove that the integral of the square of the difference $f - \varphi$ multiplied by $\overline{L}$ vanishes. We thus consider the integral

$$\int_{(D)} \overline{L}(f-\varphi)^2 d\tau = \int_{(D)} \overline{L}[(f-\varphi_n) + (\varphi_n - \varphi)]^2 d\tau$$

$$\leq 2\int_{(D)} \overline{L}(f-\varphi_n)^2 d\tau + 2\int_{(D)} \overline{L}(\varphi_n-\varphi)^2 d\tau\,,$$

where

$$\varphi_n = \sum_{k=1}^{n} a_k V_k\,.$$

Because of (91), the first summand on the right-hand side of the last inequality is less than 2ε if $n \geqq N$.

Since φ_n converges uniformly to φ, the second summand has limit zero as $n \to \infty$ and is thus less than 2ε for sufficiently large n. From this there follows the inequality:

$$\int_{(D)} \overline{L}(f - \varphi)^2 d\tau < 4\varepsilon .$$

Since however ε is arbitrary, this inequality implies that

$$\int_{(D)} \overline{L}(f - \varphi)^2 d\tau = 0 .$$

Since f and φ are continuous functions and $L > 0$, one thus obtains the result $f = \varphi$. The theorem is herewith proved since φ is the sum of a uniformly and absolutely convergent series of eigenfunctions.

Scholium. In §9 it was proved that a function $f(M)$ regular in (D) which satisfies our boundary conditions can be represented by the integral (90) with $h(M) = -\Delta f/L$. From the theorem of HILBERT and SCHMIDT we may conclude that such a function can be expanded in a uniformly and absolutely convergent series of eigenfunctions. If one accounts with the remarks of §9, then this argument also applies to continuous functions which satisfy our boundary conditions and have square-summable LAPLACE expressions which may be bounded or unbounded.

§16. The Functions of A. KORN

We consider the integral equation

$$V(0) = \frac{\lambda}{4\pi} \int_{(D_1)} \frac{L(1)\, V(1)}{r_{10}} d\tau_1 + f(0) \tag{92}$$

in which $L(0)$ is a bounded function which assumes no negative values. It was previously proved that the integral

$$\int_{(D_1)} \frac{L(1)}{r_{10}^2} d\tau_1$$

is convergent. Hence, we may apply the considerations of §§11, 12, and 13 to equation (92).

We thus obtain the result that if the sequence of eigenfunctions

$$V_1, V_2, \ldots, V_k, \ldots$$

is orthonormal, then the closedness condition

$$\frac{1}{4\pi} \int_{(D)} L(0)\, [f(0)]^2 d\tau = \sum_{k=1}^{\infty} a_k^2 , \qquad a_k = \frac{1}{4\pi} \int_{(D)} L(0) f(0) V_k(0)\, d\tau$$

holds for every function $f(0)$ which is given by the equation

$$f(0) = \frac{1}{4\pi} \int_{(D_1)} \frac{L(1)\, h(1)}{r_{10}} d\tau_1 . \tag{93}$$

The functions defined by equation (93) are Newtonian potentials. If $h(0)$ is bounded they and their first derivatives are continuous everywhere and outside the region (D) they satisfy the LAPLACE equation.

If $L(0)h(0)$ is H-continuous in (D), then in this region

$$\Delta f = -L(0)h(0).$$

In the following we shall assume that the function $L(0)$ is H-continuous.

If $V_k(0)$ is one of the eigenfunctions, then

$$V_k(0) = \frac{\lambda_k}{4\pi} \int\limits_{(D_i)} \frac{L(1)\, V_k(1)}{r_{10}}\, d\tau_1 .$$

From this equation it follows that V_k has first derivatives everywhere and is thus H-continuous in (D).

Hence, every eigenfunction V_k and its first derivatives are continuous in all of space; the function has second derivatives, and in the interior of (D)

$$\Delta V_k = -\lambda_k L V_k;$$

outside (D)

$$\Delta V_k = 0.$$

Multiplying both sides of the next to the last equation by V_k and integrating over (D_i), we obtain:

$$\begin{aligned}
-\lambda_k \int\limits_{(D_i)} L V_k^2 \, d\tau &= \int\limits_{(D_i)} V_k \Delta V_k \, d\tau = \int\limits_{(D_i)} \sum \frac{\partial\left(V_k \frac{\partial V_k}{\partial x}\right)}{\partial x}\, d\tau - \int\limits_{(D_i)} \sum \left(\frac{\partial V_k}{\partial x}\right)^2 d\tau \\
&= \int\limits_{(S)} V_k \frac{d V_{ki}}{d n}\, d\sigma - \int\limits_{(D_i)} \sum \left(\frac{\partial V_k}{\partial x}\right)^2 d\tau = \int\limits_{(S)} V_k \frac{d V_{ke}}{d n}\, d\sigma - \int\limits_{(D_i)} \sum \left(\frac{\partial V_k}{\partial x}\right)^2 d\tau \\
&= -\int\limits_{(D_e)} \sum \frac{\partial\left(V_k \frac{\partial V_k}{\partial x}\right)}{\partial x}\, d\tau - \int\limits_{(D_i)} \sum \left(\frac{\partial V_k}{\partial x}\right)^2 d\tau \\
&= -\int\limits_{(D_e)} \sum \left(\frac{\partial V_k}{\partial x}\right)^2 d\tau - \int\limits_{(D_i)} \sum \left(\frac{\partial V_k}{\partial x}\right)^2 d\tau .
\end{aligned}$$

From this it follows that all eigenvalues of equation (92) are positive.

The arguments presented in §14 must here be modified somewhat. Let ψ be a Newtonian potential defined in (D):

$$\psi = \int\limits_{(D)} \frac{L(1)\, d\tau_1}{r_{10}} .$$

After we have constructed the polynomial P which differs from the continuous function $f(0)$ in (D) by less than ε,

$$|f - P| < \varepsilon,$$

we form a function F_1 with the following properties: (a) F_1 is equal to P in a region (D') which is so chosen that the volume of the region $(D - D')$ is less than ε; (b) in $(D - D')$ the function F_1 is such that in (D) it has continuous first derivatives, and in $(D - D')$ it has bounded second derivatives; (c) on the boundary (S) of (D) it satisfies the conditions

$$(F_1)_i = \psi\,, \qquad \frac{dF_{1i}}{dn} = \frac{d\psi}{dn}. \tag{94}$$

Since the triples D_xF_1, D_yF_1, D_zF_1 and $D_x\psi$, $D_y\psi$, $D_z\psi$ are equal componentwise from the first of the two equations (94), the second equation gives:

$$\left(\frac{\partial F_1}{\partial x}\right)_i = \frac{\partial \psi}{\partial x}\,, \qquad \left(\frac{\partial F_1}{\partial y}\right)_i = \frac{\partial \psi}{\partial y}\,, \qquad \left(\frac{\partial F_1}{\partial z}\right)_i = \frac{\partial \psi}{\partial z}\,.$$

We now form the function F by putting $F = F_1$ in the interior of (D) and $F = \psi$ outside (D). The function F has continuous first derivatives everywhere, it has second derivatives in the interior of (D), and outside (D) it satisfies the LAPLACE equation.

From the considerations of II, §17 it thus follows that

$$F = -\frac{1}{4\pi}\int\limits_{(D_1)} \frac{\Delta F}{r_{10}}\, d\tau_1 = -\frac{1}{4\pi}\int\limits_{(D_1)} \frac{L(1)}{r_{10}}\,\frac{\Delta F}{L(1)}\, d\tau_1\,.$$

Hence, the function F belongs to the class of functions (93) for which the closedness of the sequence of eigenfunctions has been proved.

Repeating the considerations of §14, we obtain for the space of continuous functions the inequality

$$\int\limits_{D)} L(0)\,[f(0) - F(0)]^2 d\tau < a\,\varepsilon$$

and the equation

$$\frac{1}{4\pi}\int\limits_{(D)} L(0)[f(0)]^2 d\tau = \sum_{k=1}^{\infty} b_k^2\,, \qquad b_k = \frac{1}{4\pi}\int\limits_{(D)} L(0) f(0)\, V_k(0)\, d\tau\,.^{1)} \tag{95}$$

Without going into the proof of equation (95) for the space of bounded and integrable functions, we shall now turn to an application of the theorem of HILBERT and SCHMIDT presented in §15: *Every Newtonian potential*

$$f(0) = \frac{1}{4\pi}\int\limits_{(D_1)} \frac{L(1)\, h(1)}{r_{10}}\, d\tau_1\,,$$

in which $h(0)$ is a square-integrable function can be expanded in an absolutely and uniformly convergent series

$$\sum_{k=1}^{\infty} a_k V_k(0)$$

with

$$a_k = \frac{1}{4\pi}\int\limits_{(D)} L(0) f(0)\, V_k(0)\, d\tau = \frac{1}{4\pi\lambda_k}\int\limits_{(D)} L(0)\, h(0)\, V_k(0)\, d\tau\,.$$

The functions $V_k(0)$ are called *the universal functions of* A. KORN.

§17. Integration of the Wave Equation

In conclusion we shall study the equation

$$\frac{\partial^2 U}{\partial t^2} = a^2 \left\{ \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right\} + \Phi(x, y, z), \quad a^2 = \text{const.}, \tag{96}$$

It is desired to find a solution U which on (S) satisfies one of the three conditions

$$\left.\begin{array}{ll} (\alpha) & U = \varphi, \\ (\beta) & \dfrac{dU_i}{dn} + hU = \varphi, \\ (\gamma) & \dfrac{dU_i}{dn} = \varphi \end{array}\right| \tag{97}$$

where φ is a function of the points of (S) which may depend on the time t; in addition, the initial conditions

$$U = f_1(x, y, z), \qquad \frac{\partial U}{\partial t} = F_1(x, y, z) \quad \text{at} \quad t = 0. \tag{98}$$

are to be satisfied.

In case (γ) these conditions determine the motion of a gas enclosed in a vessel; the velocity of the gas has a potential. The function U is here the velocity potential, and if the vessel is rigid $\varphi = 0$.

To make use of the results of §10, we make the substitution

$$U = V + U_1$$

and replace conditions (97) by the conditions

$$\left.\begin{array}{ll} (\alpha) & V = 0 \\ (\beta) & \dfrac{dV_i}{dn} + hV = 0 \\ (\gamma) & \dfrac{dV_i}{dn} = 0 \end{array}\right| \quad \text{on } (S).$$

Equation (96) now assumes the form

$$\begin{aligned} \frac{\partial^2 V}{\partial t^2} &= a^2 \left\{ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right\} + \Phi + a^2 \Delta U_1 - \frac{\partial^2 U_1}{\partial t^2} \\ &= a^2 \Delta V + K(x, y, z, t); \end{aligned} \tag{99}$$

in cases (α) and (β) the expression ΔU_1 is equal to zero; in case (γ) it is equal to

$\frac{1}{D}\int\limits_{(S)}\varphi d\sigma$. In cases ($\alpha$) and ($\beta$) the functions $K(x,y,z,t)$ and Φ are distinct only if φ is time-dependent.

The initial conditions for V are

$$V = f(x, y, z), \qquad \frac{\partial V}{\partial t} = F(x, y, z) \quad \text{at} \quad t = 0 \tag{100}$$

with

$$f = f_1 - U_1(x, y, z, 0), \qquad F = F_1 - \left(\frac{\partial U_1}{\partial t}\right)_0.$$

The problem thus reduces to integrating equation (99) with the boundary conditions

$$\left.\begin{array}{ll}(\alpha) & V = 0\\ (\beta) & \frac{dV_i}{dn} + hV_i = 0\\ (\gamma) & \frac{dV_i}{dn} = 0\end{array}\right| \quad \text{on } (S)$$

and the initial conditions (100):

$$V = f(x, y, z), \qquad \frac{\partial V}{\partial t} = F(x, y, z) \quad \text{at} \quad t = 0.$$

Remark. The method of solution which we employ can also be applied to the equation

$$\frac{\partial^2 V}{\partial t^2} = a^2 \Delta V + K(x, y, z, t) + l \frac{\partial V}{\partial t}.$$

For brevity of presentation, however, we shall study only equation (99).

It is assumed that the problem in question has a solution V with continuous second derivative $\frac{\partial^2 V}{\partial t^2}$ in (D) for all t satisfying the inequality $t \geq 0$. If we apply to equation (99) the method used in treating equation (42) in §9 we can conclude that

$$V = -\frac{1}{4\pi a^2}\int\limits_{(D_1)} G(1, 0)\left(\frac{\partial^2 V}{\partial t^2} - K\right) d\tau_1; \tag{101}$$

$G(1,0)$ is here the GREEN's function appropriate to the boundary condition, e.g., in case (γ) it is the modified NEUMANN function. But if now the function V is given by equation (101), then one sees from the theorem of HILBERT and SCHMIDT that it can be expanded in a uniformly and absolutely convergent series of eignfunctions of the integral equation with kernel $G(1,0)$. From this follows: If the second derivative of the solution with respect to t is continuous, then

$$V = \sum_{k=1}^{\infty} g_k(t) V_k(0), \qquad g_k(t) = \int\limits_{(D)} V(0) V_k(0) d\tau, \tag{102}$$

where the series converges absolutely and uniformly.

The coefficients in this series must be computed. For this purpose we write equation (101) in the form

$$V = -\frac{1}{4\pi a^2}\int\limits_{(D_1)} G(1,0)\frac{\partial^2 V}{\partial t^2}\,d\tau_1 + \frac{1}{4\pi a^2}\int\limits_{(D_1)} G(1,0)K(1)\,d\tau_1$$

and integrate it twice with respect to t, taking conditions (100) into account. We obtain

$$\left.\begin{aligned}\int_0^t V\,d\zeta &= -\frac{1}{4\pi a^2}\int\limits_{(D_1)} G(1,0)\left[\frac{\partial V}{\partial t} - F(1)\right]d\tau_1\\ &\quad + \frac{1}{4\pi a^2}\int_0^t\left[\int\limits_{(D_1)} G(1,0)K(1)\,d\tau_1\right]d\zeta,\\ \int_0^t\int_0^\zeta V(\eta)\,d\eta\,d\zeta &= -\frac{1}{4\pi a^2}\int\limits_{(D_1)} G(1,0)V(1)\,d\tau_1\\ &\quad + \frac{1}{4\pi a^2}\int\limits_{(D_1)} G(1,0)f(1)\,d\tau_1\\ &\quad + \frac{t}{4\pi a^2}\int\limits_{(D_1)} G(1,0)F(1)\,d\tau_1\\ &\quad + \frac{1}{4\pi a^2}\int_0^t\int_0^\zeta\left[\int\limits_{(D_1)} G(1,0)K(1)\,d\tau_1\right]d\eta\,d\zeta.\end{aligned}\right\} \tag{103}$$

We put

$$a_k = \int\limits_{(D)} f V_k\,d\tau, \qquad b_k = \int\limits_{(D)} F V_k\,d\tau, \qquad c_k(t) = \int\limits_{(D)} K V_k\,d\tau.$$

The formulas of §15 then give:

$$\left.\begin{aligned}\int\limits_{(D_1)} G(1,0)f(1)\,d\tau_1 &= \sum_{k=1}^{\infty}\frac{a_k}{\lambda_k}V_k,\\ \int\limits_{(D_1)} G(1,0)F(1)\,d\tau_1 &= \sum_{k=1}^{\infty}\frac{b_k}{\lambda_k}V_k,\\ \int\limits_{(D_1)} G(1,0)K(1)\,d\tau_1 &= \sum_{k=1}^{\infty}\frac{c_k(t)}{\lambda_k}V_k.\end{aligned}\right\} \tag{104}$$

Each of these series converges absolutely and uniformly. The last series, considered as a function of t, is uniformly convergent; one obtains an estimate of its remainder by passing to the limit $m \to \infty$ in the following inequality:

$$\left(\sum_{k=n}^{m} \frac{c_k(t)\,V_k}{\lambda_k}\right)^2 \leq \sum_{k=n}^{m} c_k^2(t) \cdot \sum_{k=n}^{m} \frac{V_k^2}{\lambda_k^2} < \int_{(D)} K^2 d\tau \cdot \varepsilon, \quad \text{for} \quad n \geq N.$$

The index N is independent of t; it is determined by the properties of the series

$$\sum_{k=1}^{\infty} \frac{V_k^2}{\lambda_k^2}$$

whose terms are the squares of the FOURIER coefficients corresponding to the function $G(1,0)$. The same can be said of the series (102) which converges uniformly as a function of t; the coefficient $g_k(t)$ is equal to $\frac{h_k(t)}{\lambda_k}$, where by $h_k(t)$ one means the FOURIER coefficients belonging to the function

$$-\frac{1}{4\pi a^2}\left(\frac{\partial^2 V}{\partial t^2} - K\right).$$

Making use of the uniform convergence of series (102), it follows from (103):

$$\sum_{k=1}^{\infty} \int_0^t \int_0^\zeta g_k(\eta)\, d\eta\, d\zeta \cdot V_k$$

$$= -\frac{1}{4\pi a^2} \sum_{k=1}^{\infty} \frac{g_k(t)}{\lambda_k} V_k + \frac{1}{4\pi a^2} \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k} V_k$$

$$+ \frac{t}{4\pi a^2} \sum_{k=1}^{\infty} \frac{b_k}{\lambda_k} V_k + \frac{1}{4\pi a^2} \sum_{k=1}^{\infty} \frac{1}{\lambda_k} \int_0^t \int_0^\zeta c_k(\eta)\, d\eta\, d\zeta \cdot V_k.$$

Because of the orthonormality of the eigenfunctions V_k and the uniform convergence of all series, the following equation is valid:

$$4\pi a^2 \lambda_k \int_0^t \int_0^\zeta g_k(\eta)\, d\eta\, d\zeta = -g_k(t) + a_k + t b_k + \int_0^t \int_0^\zeta c_k(\eta)\, d\eta\, d\zeta.$$

From this it follows that

$$g_k(0) = a_k; \tag{105}$$

differentiating once, we find:

$$g_k'(0) = b_k; \tag{106}$$

on differentiating twice it follows that

$$g_k''(t) + 4\pi a^2 \lambda_k g_k(t) = c_k(t). \tag{107}$$

One thus obtains $g_k(t)$ by integrating (107) with conditions (105) and (106).

The general solution of equation (107) is

$$g_k(t) = c_1 \cos 2a\sqrt{\pi\lambda_k}\,t + c_2 \sin 2a\sqrt{\pi\lambda_k}\,t + \frac{1}{2a\sqrt{\pi\lambda_k}}\int_0^t c_k(\zeta)\sin 2a\sqrt{\pi\lambda_k}(t-\zeta)\,d\zeta\,.$$

From this equation we see that the solution sought has the following form:

$$g_k(t) = a_k \cos 2a\sqrt{\pi\lambda_k}\,t + \frac{b_k}{2a\sqrt{\pi\lambda_k}}\sin 2a\sqrt{\pi\lambda_k}\,t + \frac{1}{2a\sqrt{\pi\lambda_k}}\int_0^t c_k(\zeta)\sin 2a\sqrt{\pi\lambda_k}(t-\zeta)\,d\zeta\,.$$

Hence, a solution V of the problem in question with continuous second derivative $\frac{\partial^2 V}{\partial t^2}$ for all t satisfying the inequality $t \geqq 0$ can only be given by the series

$$V = \sum_{k=1}^{\infty}\left\{a_k \cos 2a\sqrt{\pi\lambda_k}\,t + \frac{b_k}{2a\sqrt{\pi\lambda_k}}\sin 2a\sqrt{\pi\lambda_k}\,t\right\}V_k + \sum_{k=1}^{\infty}\frac{V_k}{2a\sqrt{\pi\lambda_k}}\int_0^t c_k(\zeta)\sin 2a\sqrt{\pi\lambda_k}(t-\zeta)\,d\zeta \tag{108}$$

which must converge uniformly. If this series does not converge uniformly, then there is no solution of the problem in which V has a continuous second derivative with respect to t in (D).[2]

For the case in which the series (108) converges absolutely and uniformly we introduce the expression

$$R[\varphi(t)] = \frac{\varphi(t+2h) - 2\varphi(t+h) + \varphi(t)}{h^2}$$

If

$$\Phi(t) = \int_0^t\int_0^\zeta \varphi(\eta)\,d\eta\,d\zeta\,,$$

then one easily recognizes the validity of the following equation:

$$R[\Phi(t)] = \frac{1}{h^2}\int_t^{t+h}\left(\int_\zeta^{\zeta+h}\varphi(\eta)\,d\eta\right)d\zeta\,.$$

Indeed,

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[2] The results obtained in §15 afford the following conclusion: If the series (108) does not converge uniformly, then the problem has no solutions whose second derivatives with respect to t are square-integrable.

$$R[\Phi(t)] = \frac{1}{h^2}\left[\int_0^{t+2h}\left(\int_0^{\zeta}\varphi(\eta)\,d\eta\right)d\zeta - 2\int_0^{t+h}\left(\int_0^{\zeta}\varphi(\eta)\,d\eta\right)d\zeta + \int_0^{t}\left(\int_0^{\zeta}\varphi(\eta)\,d\eta\right)d\zeta\right]$$

$$= \frac{1}{h^2}\left[\int_{t+h}^{t+2h}\left(\int_0^{\zeta}\varphi(\eta)\,d\eta\right)d\zeta - \int_t^{t+h}\left(\int_0^{\zeta}\varphi(\eta)\,d\eta\right)d\zeta\right].$$

If we replace ζ by $\zeta + h$ in the first integral, then we obtain:

$$R[\Phi(t)] = \frac{1}{h^2}\left[\int_t^{t+h}\left(\int_0^{\zeta+h}\varphi(\eta)\,d\eta\right)d\zeta - \int_t^{t+h}\left(\int_0^{\zeta}\varphi(\eta)\,d\eta\right)d\zeta\right]$$

$$= \frac{1}{h^2}\int_t^{t+h}\left(\int_{\zeta}^{\zeta+h}\varphi(\eta)\,d\eta\right)d\zeta\,.$$

If we replace φ by $V(t)$ and recall that the function defined by the series (108) satisfies equations (103), then we find on forming the expression R for the second equation (103):

$$\frac{1}{h^2}\int_t^{t+h}\left(\int_{\zeta}^{\zeta+h}V(\eta)\,d\eta\right)d\zeta = -\frac{1}{4\pi a^2}\int_{(D_1)}G(1,0)\,R[V(t)]\,d\tau_1$$
$$+\frac{1}{4\pi a^2}\cdot\frac{1}{h^2}\int_t^{t+h}\left[\int_{\zeta}^{\zeta+h}\left(\int_{(D_1)}G(1,0)\,K(1)\,d\tau_1\right)d\eta\right]d\zeta\,.$$

If $V(t)$ has a second derivative with respect to t, then clearly

$$\lim_{h\to 0} R[V(t)] = \frac{\partial^2 V}{\partial t^2}\cdot$$

If the expression $R[V(t)]$ converges uniformly in (D), then

$$\lim_{h\to 0}\int_{(D_1)}G(1,0)\,R[V(t)]\,d\tau_1 = \int_{(D_1)}G(1,0)\lim_{h\to 0}R[V(t)]\,d\tau_1\,.$$

Since $\varphi(t)$ is the limit of

$$\frac{1}{h^2}\int_t^{t+h}\left(\int_{\zeta}^{\zeta+h}\varphi(\eta)\,d\eta\right)d\zeta$$

for $h \to 0$, we arrive at the following result: If the expression

$$R[V(t)]$$

for $t \geqq 0$ converges uniformly to its limit as $h \to 0$, then the sum of the series (108) satisfies equation (101). If moreover the function

$$\frac{\partial^2 V}{\partial t^2} - K$$

is H-continuous in (D), then the function V satisfies equation (99) and hence provides the solution of the problem.

To see that the initial conditions (100) are satisfied, one need only recall that the series (108) satisfies the second of equations (103). Integrating equation (101) twice with respect to t and subtracting the second of equations (103) from the result, we obtain for all t satisfying the inequality $t \geqq 0$:

$$t \int_{(D_1)} G(1, 0) \left[\left(\frac{\partial V}{\partial t} \right)_{t=0} - F(1) \right] d\tau_1 + \int_{(D_1)} G(1, 0) [V_{t=0} - f(1)] d\tau_1 = 0 .$$

It now follows from the completeness of the kernel $G(M,M')$ proved in §11 that

$$V_{t=0} = f(0), \qquad \left(\frac{\partial V}{\partial t} \right)_{t=0} = F(0) .$$ [1]

We shall not here investigate the conditions under which the series (108) converges uniformly.

Remark. It has recently been possible to make strong assertions regarding the solution (108) of the wave equation (99).

To have a specific case in mind, let us consider the problem with condition (α). Suppose that the boundary (S) of the region belongs to the class L_5. Let further the functions $f(x,y,z)$ and $F(x,y,z)$ occurring in the initial conditions (100) have continuous derivatives up to fourth and third order respectively, and on (S) let them satisfy the following conditions:

$$f = \Delta f = F = \Delta F = 0 \quad .$$

Moreover, let $K = 0$. Under these conditions it is possible to prove that series (108) and the series obtained from it after differentiating twice with respect to x, y, z, and t converge uniformly in (D) for $t \geqq 0$. The series (108) then evidently represents the twice differentiable solution of the problem. These results are found in the paper of O. A. LADYZHENSKAYA, "O metode fur'e dlya volnovogo uravneniya" ("On Fourier Methods for the Wave Equation"), Doklady AN SSSR, Vol. 75, No. 6, 1950. The detailed proof of the analogous assertion for the wave equation in three independent variables x,y,t can be found in Vol. IV of the book *A Course of Higher Mathematics* by V. I. SMIRNOV.

§18. On the Heat Problem

As a second example we treat the heat problem. We showed in §7 that the amount of heat passing out of a region (ω) in the interior of a body (D) in time dt through the boundary (σ) of (ω) is equal to

$$dt \cdot k \int\limits_{(\sigma)} \frac{du}{dn}\, d\sigma \; ;$$

u is here the temperature, and k is the thermal conductivity. Hence, the amount of heat delivered to the region (ω) is equal to

$$dt \cdot k \int\limits_{(\sigma)} \frac{du}{dn}\, d\sigma \, .$$

On the other hand, this quantity is proportional to the increase in temperature

$$u(t + dt) - u(t) = \frac{\partial u}{\partial t}\, dt \, ,$$

where the proportionality factor is equal to the product of the specific heat of the body and the amount of mass $\varrho\omega$ in (ω).

Since heat sources may be present in (ω), we have:

$$\frac{\partial u}{\partial t}\, c\varrho\omega\, dt = dt \cdot k \int\limits_{(\sigma)} \frac{du}{dn}\, d\sigma + F_1\, dt \cdot \omega \, .$$

Dividing by $c\varrho\omega dt$ and then letting ω go to zero, we obtain the following equation:

$$\frac{\partial u}{\partial t} = a^2 \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right\} + F, \qquad a^2 = \frac{k}{\varrho c} \, . \tag{109}$$

On the boundary the temperature must satisfy one of the conditions

$$\left.\begin{array}{ll} (\alpha) & u = \varphi \, , \\ (\beta) & \dfrac{du_i}{dn} + hu = \varphi \, , \\ (\gamma) & \dfrac{du_i}{dn} = \varphi \end{array}\right| \tag{110}$$

according to whether the surface is held at a particular temperature, radiation takes place into the space surrounding the body, or finally the heat loss on the boundary is given.

If the temperature distribution is given initially, then we also have the initial condition

$$u = f_1(x,y,z) \quad \text{for} \quad t = 0 \, . \tag{111}$$

Making use again of the results in §10, we make the substitution

$$u = V + u_1 ,$$

where u_1 is appropriately chosen. As in the preceding section, we transform equation (109) to the form

$$\frac{\partial V}{\partial t} = a^2 \left\{ \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \right\} + K . \tag{109'}$$

Conditions (110) become conditions

$$\left.\begin{aligned} &(\alpha) & V &= 0 , \\ &(\beta) & \frac{d V_i}{d n} + h V &= 0 , \\ &(\gamma) & \frac{d V_i}{d n} &= 0 , \end{aligned}\right| \tag{110'}$$

while for the initial condition one can write:

$$V = f(x,y,z) \quad \text{at } t = 0 . \tag{111'}$$

If now as in §17 we apply the method of §9 to equation (109′), we find:

$$V = -\frac{1}{4\pi a^2} \int_{(D_1)} G(1,0) \left(\frac{\partial V}{\partial t} - K \right) d\tau_1 ; \tag{112}$$

$G(1,0)$ is here the appropriate GREEN's function.

Now if the function V is given by equation (112), then it can be expanded in terms of the eigenfunctions of the integral equation with kernel $G(1,0)$. The series thus obtained is absolutely and uniformly convergent under the sole condition that the function $\frac{\partial V}{\partial t} - K$ is continuous in (D).[3]

Let

$$V = \sum_{k=1}^{\infty} g_k(t)\, V_k(0) \tag{113}$$

be this series. To compute the coefficients, we integrate both sides of equation (112) with respect to t. We obtain:

[3] According to the results of §15, this series is also absolutely and uniformly convergent if $\frac{\partial V}{\partial t} - K$ is only square-integrable.

$$\int_0^t V\,d\zeta = -\frac{1}{4\pi a^2}\int_{(D_1)} G(1,0)(V-f)\,d\tau_1 + \frac{1}{4\pi a^2}\int_0^t\left(\int_{(D_1)} G(1,0)\,K(1)\,d\tau_1\right)d\zeta\,. \tag{114}$$

Putting

$$a_k = \int_{(D)} f\,V_k\,d\tau\,, \qquad c_k(t) = \int_{(D)} K\,V_k\,d\tau\,,$$

we find:

$$\int_{(D_1)} G(1,0)f(1)\,d\tau_1 = \sum_{k=1}^{\infty}\frac{a_k}{\lambda_k}V_k\,,$$

$$\int_{(D_1)} G(1,0)\,K(1)\,d\tau_1 = \sum_{k=1}^{\infty}\frac{c_k(t)}{\lambda_k}V_k\,.$$

Now putting the series (113) into (114) for V, we obtain:

$$\sum_{k=1}^{\infty}V_k\int_0^t g_k(\zeta)\,d\zeta = -\frac{1}{4\pi a^2}\sum_{k=1}^{\infty}\frac{g_k(t)V_k}{\lambda_k} + \frac{1}{4\pi a^2}\sum_{k=1}^{\infty}\frac{a_kV_k}{\lambda_k} + \frac{1}{4\pi a^2}\sum_{k=1}^{\infty}\frac{V_k}{\lambda_k}\int_0^t c_k(\zeta)\,d\zeta\,.$$

Hence,

$$g_k(0) = a_k \tag{115}$$

and

$$g_k'(t) + 4\pi a^2\lambda_k g_k(t) = c_k(t)\,. \tag{116}$$

Integrating (116) subject to condition (115), we find:

$$g_k(t) = a_k e^{-4\pi a^2\lambda_k t} + e^{-4\pi a^2\lambda_k t}\int_0^t c_k(\zeta)\,e^{4\pi a^2\lambda_k\zeta}\,d\zeta\,.$$

Thus, the series (113) has the form

$$V = \sum_{k=1}^{\infty}\left\{a_k e^{-4\pi a^2\lambda_k t} + e^{-4\pi a^2\lambda_k t}\int_0^t c_k(\zeta)\,e^{4\pi a^2\lambda_k\zeta}\,d\zeta\right\}V_k\,. \tag{117}$$

If series (117) does not converge uniformly in (D) for $t > 0$, then the problem has no solutions in which V has a continuous derivative with respect to t for $t \geq 0$[4].

Suppose now that the series (117) converges absolutely and uniformly. We introduce the expression

[4] The problem then also has no solutions in which $\frac{\partial V}{\partial t} - K$ is a square-integrable function.

$$R[\varphi(t)] = \frac{\varphi(t+h) - \varphi(t)}{h}$$

If

$$\Phi(t) = \int_0^t \varphi(\zeta)\,d\zeta,$$

then clearly

$$R[\Phi(t)] = \frac{1}{h}\int_t^{t+h} \varphi(\zeta)\,d\zeta.$$

Since the series (117) satisfies equation (114), we find on forming the expression R and replacing φ by $V(t)$ that

$$\begin{aligned}\frac{1}{h}\int_t^{t+h} V(\zeta)\,d\zeta = &-\frac{1}{4\pi a^2}\int_{(D_1)} G(1,0)\,R[V(t)]\,d\tau_1\\ &+\frac{1}{4\pi a^2}\cdot\frac{1}{h}\int_t^{t+h}\left(\int_{(D_1)} G(1,0)\,K(1)\,d\tau_1\right)d\zeta.\end{aligned} \tag{118}$$

If $V(t)$ has a first derivative with respect to t for $t \geqq 0$, then

$$\lim_{h\to 0} R[V] = \frac{\partial V}{\partial t}.$$

If the quantity $R[V]$ tends uniformly to its limit, then we obtain equation (112) immediately from equation (118); but from (112) one can conclude that equation (109′) is satisfied if $\frac{\partial V}{\partial t} - K$ is H-continuous in (D).

With the series (117) we have found the solution of equation (109′) which satisfies the boundary conditions (110′). This solution also satisfies the initial condition (111′). Integrating (112) with respect to t and subtracting the result from equation (114), we obtain:

$$\int_{(D_1)} G(1,0)\,[f(1) - V_{t=0}]\,d\tau_1 = 0.$$

From the completeness of the kernel $G(M,M')$ proved in §11 we now conclude that $V_{t=0} = f(0)$.

We shall not go into the conditions under which the series (117) converges uniformly here. We remark only the following: If $f(0)$ can be expanded in a uniformly convergent series of eigenfunctions, i.e., if the series

$$f(0) = \sum_{k=1}^{\infty} a_k V_k \tag{119}$$

converges uniformly, and if further K is equal to zero, then the function (117), which here takes the form

$$V = \sum_{k=1}^{\infty} a_k e^{-4\pi a^2 \lambda_k t} V_k, \tag{120}$$

converges uniformly for all t satisfying the inequality $t \geqq 0$ and has a derivative with respect to t for all t such that $t \geqq \alpha$ with $\alpha > 0$. In fact,

$$e^{-4\pi a^2 \lambda_k t} < 1$$

and for every $\alpha > 0$

$$4\pi a^2 \lambda_k e^{-4\pi a^2 \lambda_k t} < 1$$

for λ_k sufficiently large.

To verify that the function (120) is the solution of the problem in question, it thus suffices to show that the sum of the series

$$-4\pi a^2 \sum_{k=1}^{\infty} \lambda_k e^{-4\pi a^2 \lambda_k t} a_k V_k,$$

which is uniformly continuous in (D), is also H-continuous there.

§19. A Remark on Problems Related to the LAPLACE Operator

The fact that in §§17 and 18 we were able to prove only weak statements is due to the circumstance that the POISSON theorem may be applied only to Newtonian potentials with H-continuous densities. Somewhat more general results can be obtained by replacing equation (109′) by the equation

$$a^2 \int\limits_{(\sigma)} \frac{dV}{dn}\, d\sigma = \frac{\partial}{\partial t} \int\limits_{(\omega)} V\, d\tau - \int\limits_{(\omega)} K\, d\tau$$

and changing the boundary and initial conditions in an appropriate manner if necessary; (ω) is here an arbitrary region in the interior of (D) and (σ) is its surface.

One can, for example, require that each mean value of the temperature V at the initial time point is equal to the corresponding mean value of a given function $f(0)$; the latter mean value function can, according to the theorem of STEKLOV, always be expanded in a convergent series of mean values of eigenfunctions.

More general results can be obtained by introducing distributions and generalizing the POISSON theorem correspondingly.

§20. A Remark on the Solution of the POISSON Equation and the Eigenfunctions

We shall prove the following theorem.

Theorem 1. *If* $(S) \in L_{k+1}(B,\lambda)$ $(k \geqq 0)$ *and* $\varphi \in H(l,A,\lambda)$ $(0 \leqq l \leqq k)$ *in* (D_i), *then the solution of the equation*

$$\Delta u = -4\pi\varphi \tag{121}$$

with conditions either

$$u|_S = 0 \tag{122}$$

or

$$\left.\frac{du}{dn}\right|_S = 0 \tag{123}$$

belongs to the class $H(1 + 2,cA,\lambda')$ *if* $l < k$ *and to the class* $H(k + l,cA,\lambda')$ *if* $l = k$.

Proof. It follows from the conditions $\varphi \in H(l,A,\lambda)$ and $(S) \in L_{k+1}(B,\lambda)$ that $P[\varphi] \in H(l + 2,cA,\lambda')$ in (D_i) and in (D_e) (Theorem 2 of II, §20) and that $P[\varphi]$ satisfies equation (121) in (D_i) and the LAPLACE equation in (D_e). Hence, every solution u can be represented in the form

$$u = P[\varphi] - v \ , \tag{124}$$

where v is a harmonic function which on the boundary satisfies the condition

$$v|_S = P[\varphi] \tag{125}$$

or the condition

$$\left.\frac{dv_i}{dn}\right|_S = \frac{dP[\varphi]}{dn} \ . \tag{126}$$

We begin with the first problem. If $l < k$, then $l + 2 \leqq k + 1$, so that the limit values of $P[\varphi]$ on (S) form a function of class $H(l + 2,cA,\lambda')$. Then as solution of the inner DIRICHLET problem the function v, according to Theorem 1 of IV, §18, also belongs to the class $H(l + 2,cA,\lambda')$. From (124) it now follows that $u \in H(l + 2,c_1A,\lambda')$, wherewith the theorem is proved for $l < k$ under condition (122).

In the case $l = k$ even though $P[\varphi] \in H(l + 2,cA,\lambda')$ in (D_i) one can only say that the limit values of $P[\varphi]$ on (S) form a function of class $H(k + 1,cA,\lambda')$ on (S); from this follows the assertion of the theorem for $l = k$ and condition (122).

Since

$$\int\limits_{(D)} \Delta u \, d\tau = \int\limits_{(S)} \frac{du}{dn} d\sigma \ , \tag{127}$$

the equation

$$\int\limits_{(D)} \varphi \, d\tau = 0 \tag{128}$$

is a necessary condition for the solvability of equation (121) with boundary condition (123). If we suppose that this condition is fulfilled, then from (127) and (128) we obtain the equation

$$\int_{(S)} \frac{dP[\varphi]}{dn} d\sigma = 0 .$$

This equation is however the sufficient condition for the solvability of the inner NEUMANN problem with condition (126). Equation (128) is thus also a sufficient condition for the solvability of equation (121) with boundary condition (123).

It is clear that in the case of condition (123) there are an infinite number of solutions of equation (121) which differ by additive constants. Let u_0 be that solution which satisfies the condition

$$\int_{(D)} u_0 d\tau = 0 . \tag{129}$$

We shall show that $u_0 \in H(l + 2,cA,\lambda')$ for $l < k$ and $u_0 \in H(k + 1,cA,\lambda')$ for $l = k$. Indeed, if ϱ_1 is the density of the simple-layer potential which is equal to one on (S) and if v_1 is the solution of the NEUMANN problem which satisfies the condition

$$\int_{(S)} \varrho_1 v_1 d\sigma = 0$$

on (S), then from Theorem 1 in III, §18 and $\frac{dP}{dn} \in H(l + 1,cA,\lambda')$ for $l < k$ and $\frac{dP}{dn} \in H(k,cA,\lambda')$ for $l = k$ it follows that $v_1 \in H(l + 2,c_1A,\lambda')$ for $l < k$ and $v_1 \in H(k + 1,c_1A,\lambda')$ for $l = k$. One has

$$v = v_1 + C' ,$$

where C' is a constant.

From condition (129) $\int_{(D)} \{P[\varphi] - v_1 - C'\} d\tau = 0 ,$

whence we obtain the inequality

$$|C'| = \frac{1}{D} \left| \int_{(D)} \{P[\varphi] - v_1\} d\tau \right| < (c + c_1) A = c_2 A .$$

Thus $v = v_1 + C' \in H(l + 2,c_3A,\lambda')$ for $l < k$ and thus also $u_0 = P[\varphi] - v \in H(l + 2,cA_4,\lambda')$ for $l < k$. An analogous result is obtained for the case $l = k$. This completes the proof of the theorem.

As an application of the theorem just proved, we shall now study certain properties of the eigenfunctions of the equation $\Delta u + \lambda Lu = 0$. Let $\lambda_1, \lambda_2, \ldots, \lambda_m, \ldots$ be the system of eigenvalues and $V_1, V_2, \ldots, V_m, \ldots$ be the corresponding system of eigenfunctions of the equation

$$\Delta V + \lambda L V = 0 \quad (L > 0). \tag{130}$$

with boundary condition

$$V = 0 \quad \text{on } (S) \tag{131}$$

and normalization condition

$$\int\limits_{(D)} L V^2 d\tau = 1. \tag{132}$$

Theorem 2. *If $(S) \in L_{k+1}(B,\alpha)$ $(k \geqq 0)$ and $L \in H(k,A,\alpha)$, then the eigenfunctions V_m have bounded and continuous derivatives up to order $k+1$. The derivatives of order l $(0 \leqq l \leqq k+1)$ are hereby bounded in absolute value by numbers of the form $a_l \lambda_m^{\frac{l}{2}+1}$ if l is even and by numbers of the form $a_l \lambda_m^{\frac{l+1}{2}+1}$ if l is odd, where a_l does not depend on the index m. Moreover, the derivatives of order $k+1$ are H-continuous.*

Proof. The existence of the second derivatives of the eigenfunctions V_m at every interior point of (D_i) was proved in §10 under the hypothesis that L is H-continuous in every region which together with its boundary is contained in the interior of (D_i). One can likewise prove: If the function L has H-continuous derivatives of order j $(0 \leqq j \leqq k)$ in some region (D') which together with its boundary is contained in the interior of (D_i), then V_m has continuous derivatives of order $j+2$ in (D').

Under the hypotheses of our theorem the functions V_m have derivatives which are continuous up to order $k+2$; one cannot however make any assertions with regard to the behavior of these derivatives in a neighborhood of the boundary (S). Our objective is to prove the boundedness and H-continuity of the derivatives of V_m up to order $k+1$, as well as to show how the upper bound of the absolute value of one of these derivatives depends on the eigenvalue λ_m.

For this we consider first of all the Newtonian potential $P\left[\frac{\lambda_m}{4\pi} L V_m\right]$. From condition (132)

$$\left\|\frac{\lambda_m}{4\pi} L V_m\right\|_{L_2} = \sqrt{\int\limits_{(D)} \frac{\lambda_m^2}{(4\pi)^2} (L V_m)^2 d\tau}$$

$$< \frac{\lambda_m}{4\pi} \sqrt{A \int\limits_{(D)} L V_m^2 d\tau} = \frac{\lambda_m}{4\pi} \sqrt{A} = c_1 \lambda_m,$$

and from the theorem in II, §24 one has:

$$P\left[\frac{\lambda_m}{4\pi} L V_m\right] \in H\left(0,\, c_2 \lambda_m,\, \frac{1}{2}\right).$$

From this it follows on the basis of Theorem 1 of IV, §19 that the harmonic function v which on (S) assumes the same values as $P\left[\frac{\lambda_m}{4\pi} L V_m\right]$ belongs to the class $H(0,c_3\lambda_m,\alpha')$ in (D_i), where $\alpha' < \mathrm{Min}\,(\frac{1}{2},\alpha)$. Since

$$V_m = P\left[\frac{\lambda_m}{4\pi} LV_m\right] - v$$

it follows from this that

$$V_m \in H(0, c_4\lambda_m, \alpha'). \tag{133}$$

This proves the theorem for $l = 0$, since $|V_m| < c_4\lambda_m = a_0\lambda_m$.

We now write equation (130) in the form

$$\Delta V_m = -4\pi\left(\frac{\lambda_m}{4\pi} LV_m\right) \tag{134}$$

and assume that the assertion

$$V_m \in H\left(l, a_l\lambda_m^{q_l}, \alpha'\right) \tag{135}$$

has been established for $l \leqq k$. From (135) we then conclude that

$$\frac{\lambda_m}{4\pi} LV_m \in H\left(l, b_l\lambda_m^{q_l+1}, \alpha'\right),$$

and from Theorem 1 and equation (134) it follows that

$$V_m \in H\left(l+2, a_{l+2}\lambda_m^{q_l+1}, \alpha''\right), \quad \text{if} \quad l < k,$$

$$V_m \in H\left(k+1, a_{k+1}\lambda_m^{q_k+1}, \alpha''\right), \quad \text{if} \quad l = k.$$

But this means that

$$q_{l+2} = q_l + 1, \quad \text{if} \quad l < k, \quad \text{and} \quad q_{k+1} = q_k + 1. \tag{136}$$

From (133) it follows that $q_0 = 1$; one then finds from (136) that $q_l = 1 + \frac{l}{2}$ if l is even and $l \leqq k + 1$.

Since $(S) \in L_{k+1} (k \geqq 0)$, it is always the case that $(S) \in L_1$; we find therefore on putting $k = 0$: $q_1 = q_0 + 1 = 2$. One could also have obtained this result in another manner by proving that the first derivatives are H-continuous with exponent 1. We shall however not further consider this alternative method of proof.

Since $q_1 = 2$ and $q_{l+2} = q_l + 1$, one has $q_l = 2 + \frac{l-1}{2} = 1 + \frac{l+1}{2}$ if l is odd and $l \leqq k + 1$. This completes the proof of the theorem.

Remark. There is a similar theorem for the eigenfunctions with the boundary condition

$$\frac{dV}{dn} = 0$$

or the boundary condition

$$\frac{dV}{dn} + hV = 0 \qquad (h = \text{const.} > 0).$$

We shall not go into the proof of this theorem.

APPENDIX

§1. The Theorem of LYAPUNOV on the First Derivatives of the Simple-Layer Potential with H-Continuous Density

Theorem (LYAPUNOV). *If the density μ of a simple-layer potential*

$$V(1) = \int_{(S_2)} \mu(2) \frac{d\sigma_2}{r_{12}}$$

is H-continuous on (S), then the first derivatives of the potential are H-continuous in (D_i) and in (D_e).

Proof. Note first of all that the derivative $\frac{\partial V}{\partial x}$ of the simple-layer potential has continuous derivatives of every order and satisfies the LAPLACE equation at every interior point of the region (D_i) or the region (D_e). Moreover, $\frac{\partial V}{\partial x}$ goes to zero at infinity. Thus, $\frac{\partial V}{\partial x}$ is a harmonic function in every region which together with its boundary is contained in (D_i) or (D_e). The theorem will be proved if we show the following: (a) $\frac{\partial V}{\partial x}$ has definite limits $\frac{\partial V_i}{\partial x}$ and $\frac{\partial V_e}{\partial x}$ as the point M_1 approaches the boundary (S) while remaining in (D_i) or in (D_e); (b) these limits $\frac{\partial V_i}{\partial x}$ and $\frac{\partial V_e}{\partial x}$ are H-continuous functions on the boundary. Indeed, a function harmonic in (D_i) or in (D_e) whose limit function is H-continuous on (S) is H-continuous in (D_i) or (D_e) from Theorem 1 of IV, §18; we should mention here that the investigation of the DIRICHLET and NEUMANN problems for LYAPUNOV surfaces was carried through without the use of the theorem we are now to prove.

In order to have a specific case in mind, we restrict ourselves to the case of the region (D_e). We must then prove that $\frac{\partial V_e}{\partial x}$ exists and is H-continuous on (S). The existence and simple continuity of $\frac{\partial V_e}{\partial x}$ were proved by LYAPUNOV; he asserted the H-continuity without proof. The existence proof given below goes back to LYAPUNOV, while the proof of H-continuity is due to N. M. GÜNTER. We assume in what follows that $\lambda < 1$.

1. Let ξ, η, ζ be a coordinate system with origin at a certain point M_0 of the surface (S) in which the ζ axis coincides with the normal N_0 to the surface at the point M_0. The point $M_1(0,0,\delta)$ with $0 < \delta < \frac{d}{4}$ then lies on this normal N_0. Now we know that as $\delta \to 0$ $\frac{\partial V}{\partial \zeta}$ has the limit $\frac{dV}{dn} - 2\pi\mu_0$; moreover, from the theorem in II, §6

$$\left|\left(\frac{\partial V}{\partial \zeta}\right)_{M_1} - \frac{dV}{dn} + 2\pi\mu_0\right| < aA\delta^\lambda. \tag{1}$$

We shall now prove that as $\delta \to 0$ $\frac{\partial V}{\partial \xi}$ likewise has a limit and differs from this limit by a quantity which is not greater than $aA\delta^\lambda$ in absolute value. It is clear that for $\left(\frac{\partial V}{\partial \eta}\right)_{M_1}$ a similar assertion will hold.

Let (Σ) be the subregion of the surface (S) which lies inside a LYAPUNOV sphere about M_0 and whose projection on the (ξ,η) plane is a circle about M_0 of radius $R \geqq \frac{d}{2}$; the exact value of R will be specified later. In the following let $M(\xi,\eta,\zeta)$ denote the integration point; moreover, let r_1 and r_0 be the distances of the point M from M_1 and M_0 respectively. We have:

$$\left(\frac{\partial V}{\partial \xi}\right)_{M_1} = \int_{(S)} \mu \frac{\xi}{r_1^3}\, d\sigma = \int_{(S-\Sigma)} \mu \frac{\xi}{r_1^3}\, d\sigma + \int_{(\Sigma)} (\mu - \mu_0) \frac{\xi}{r_1^3}\, d\sigma + \mu_0 \int_{(\Sigma)} \frac{\xi}{r_1^3}\, d\sigma, \tag{2}$$

where μ_0 is the value of μ at the point M_0.

The first integral on the right-hand side of (2) is continuous in a neighborhood of the point M_0; its limit as $\delta \to 0$ is equal to the analogous integral obtained when r_1 is replaced by r_0. Moreover, this integral differs from its limit by a quantity which is not greater than a number of the form $aA\delta$ in absolute value. We can show that the integral

$$\int_{(\Sigma)} (\mu - \mu_0) \frac{\xi}{r_0^3}\, d\sigma \tag{3}$$

converges and is equal to the limit of the second integral on the right-hand side of (2) as $\delta \to 0$. Indeed, it follows from the inequalities

$$|\mu - \mu_0| < 2A\varrho^\lambda, \qquad |\xi| \leq \varrho, \qquad \frac{1}{r_0} \leq \frac{1}{\varrho}$$

that the inequality

$$\left|\mu - \mu_0\right| \frac{|\xi|}{r_0^3} < 2A\varrho^{\lambda-2}$$

holds, which ensures the convergence of the integral. If we denote by (2δ) the subregion of (Σ) lying inside the sphere of radius 2δ about M_0, then we find

further that the absolute value of the integral $\int\limits_{(2\delta)} (\mu - \mu_0) \frac{\xi}{r_0^3} d\sigma$ is not greater than $\frac{8\pi}{\lambda} A(2\delta)^\lambda$. Since $r_1 > \varrho$ we can make an analogous estimate for the integral over (2δ) which is obtained if r_0 is replaced by r_1 in the last integral. Clearly,

$$\left|\int\limits_{(\Sigma)} (\mu - \mu_0) \frac{\xi}{r_1^3} d\sigma - \int\limits_{(\Sigma)} (\mu - \mu_0) \frac{\xi}{r_0^3} d\sigma\right|$$

$$\leq \int\limits_{(\Sigma - 2\delta)} |\mu - \mu_0| \cdot |\xi| \left|\frac{1}{r_1^3} - \frac{1}{r_0^3}\right| d\sigma + \left|\int\limits_{(2\delta)} (\mu - \mu_0) \frac{\xi}{r_1^3} d\sigma\right| \tag{4}$$

$$+ \left|\int\limits_{(2\delta)} (\mu - \mu_0) \frac{\xi}{r_0^3} d\sigma\right|.$$

The sum of the last two summands on the right-hand side of (4) is not greater than $16\pi\lambda^{-1}A(2\delta)^\lambda$. To estimate the first integral on the right-hand side of (4), note that according to inequality (9) of II, §2 the inequality $\frac{1}{r_1} < \frac{2}{r_0} \leqq \frac{2}{\varrho}$ holds if M lies on $(\Sigma - 2\delta)$. Hence,

$$\left|\frac{1}{r_1^3} - \frac{1}{r_0^3}\right| = \frac{|r_1 - r_0|}{r_1 r_0}\left(\frac{1}{r_1^2} + \frac{1}{r_1 r_0} + \frac{1}{r_0^2}\right) < \frac{14\delta}{r_0^4} \leq \frac{14\delta}{\varrho^4},$$

and therefore the absolute value of the integral in question is not greater than

$$112\pi A\delta \int\limits_\delta^d \varrho^\lambda \varrho \cdot \frac{1}{\varrho^4} \varrho \, d\varrho = 112\pi A\delta \int\limits_\delta^d \varrho^{\lambda-2} d\varrho$$

$$= 112\pi A\delta\left(\frac{\delta^{\lambda-1}}{1-\lambda} - \frac{d^{\lambda-1}}{1-\lambda}\right) < \frac{112\pi}{1-\lambda} A\delta^\lambda.$$

From this it follows that the second integral on the right-hand side of (2) differs from its limit (3) by a quantity which is less than a number of the form $aA\delta^\lambda$ in absolute value.

We now turn to the investigation of the third integral on the right-hand side of (2). First note that

$$\int\limits_0^{2\pi}\int\limits_0^R \frac{\xi}{(\sqrt{\varrho^2 + \delta^2})^3} \varrho \, d\varrho \, d\varphi = \int\limits_0^{2\pi}\int\limits_0^R \frac{\varrho^2 \cos\varphi}{(\sqrt{\varrho^2 + \delta^2})^3} d\varrho \, d\varphi = 0,$$

i.e., the equation

$$\int\limits_{(\Sigma)} \frac{\xi \cos(NN_0)}{(\sqrt{\varrho^2 + \delta^2})^3} d\sigma = 0$$

holds.

We introduce the notation $\cos(NN_0) = \gamma$. Then

$$\int\limits_{(\Sigma)} \frac{\xi}{r_1^3}\, d\sigma = \int\limits_{(\Sigma)} \xi\left(\frac{1}{r_1^3} - \frac{\gamma}{(\sqrt{\varrho^2+\delta^2})^3}\right) d\sigma = \int\limits_{(\Sigma)} \left(\frac{T^3}{\gamma} - 1\right)\frac{\xi\gamma}{(\sqrt{\varrho^2+\delta^2})^3}\, d\sigma \tag{5}$$

with

$$T = \frac{\sqrt{\varrho^2+\delta^2}}{r_1} = \frac{\sqrt{\varrho^2+\delta^2}}{\sqrt{\varrho^2+(\delta-\zeta)^2}}$$

The expression $\sqrt{\varrho^2 + \delta^2}$ is obviously the distance of the point M_1 from the projection M' of the point M on the (ξ,η) plane. It thus follows from considering the triangle M_1MM' that

$$\left|\sqrt{\varrho^2+\delta^2} - r_1\right| < |M\,M'| = |\zeta| < b\,\varrho^{1+\lambda}$$

and hence

$$|T-1| = \frac{\left|\sqrt{\varrho^2+\delta^2} - r_1\right|}{r_1} < b\,\varrho^{\lambda}. \tag{6}$$

If we choose d so small that $bd^\lambda < \frac{1}{2}$, then from this we obtain the inequality

$$\frac{1}{2} < T < \frac{3}{2}\,. \tag{6'}$$

Since $\gamma = \cos(NN_0) > \frac{1}{2}$ and

$$1-\gamma = 1 - \cos(NN_0) \leq \tfrac{1}{2}(NN_0)^2 \leq (NN_0) < E(2\varrho)^\lambda\,,$$

we find:

$$\begin{aligned}\left|\frac{T^3}{\gamma} - 1\right| &= \left|(T-1)(T^2+T+1) + (1-\gamma)\right| \cdot \frac{1}{\gamma}\\ &< 2\left[b\,\varrho^\lambda \cdot \left(\frac{9}{4} + \frac{3}{2} + 1\right) + E(2\varrho)^\lambda\right] = C\varrho^\lambda.\end{aligned} \tag{6''}$$

This estimate does not depend on δ and is thus valid for any location of the point M_1.

Thus the integrand in the integral (5) is no greater than the function

$$C\varrho^\lambda \cdot \frac{\varrho}{\varrho^3} = C\varrho^{\lambda-2}$$

in absolute value, whence it follows that the integral on the right-hand side of (5) also converges for $\delta = 0$.

We introduce the notation

$$T_0 = \frac{\varrho}{r_0} = \sin(r_0 N_0) \tag{7}$$

and wish to show that the integral

$$\int\limits_{(\Sigma)} \left(\frac{T_0^3}{\gamma} - 1\right)\frac{\xi\gamma}{\varrho^3}\, d\sigma\,, \tag{8}$$

which is the limit of integral (5) as $\delta \to 0$, differs from (5) by a quantity which is less than a number of the form $a\delta^\lambda$ in absolute value. Indeed, in one integrates the integrands in (5) and (8) over the subregion (σ) of (Σ) whose projection on the (ξ,η) plane is the circle of radius 2δ about M_0, then one obtains values which are less than numbers of the form $c\delta^\lambda$ in absolute value. It thus remains to investigate the following integral:

$$\begin{aligned}&\int\limits_{(\Sigma-\sigma)}\gamma\xi\left\{\left(\frac{T^3}{\gamma}-1\right)\frac{1}{(\sqrt{\varrho^2+\delta^2})^3}-\left(\frac{T_0^3}{\gamma}-1\right)\frac{1}{\varrho^3}\right\}d\sigma\\&=\int\limits_{(\Sigma-\sigma)}\gamma\xi\left\{\left(\frac{T_0^3}{\gamma}-1\right)\left(\frac{1}{(\sqrt{\varrho^2+\delta^2})^3}-\frac{1}{\varrho^3}\right)+\frac{1}{(\sqrt{\varrho^2+\delta^2})^3}\,\frac{T^3-T_0^3}{\gamma}\right\}d\sigma.\end{aligned}\tag{9}$$

Clearly,

$$\left|\frac{1}{(\sqrt{\varrho^2+\delta^2})^3}-\frac{1}{\varrho^3}\right|=\frac{\sqrt{\varrho^2+\delta^2}-\varrho}{\varrho\sqrt{\varrho^2+\delta^2}}\left(\frac{1}{\varrho^2}+\frac{1}{\varrho\sqrt{\varrho^2+\delta^2}}+\frac{1}{\varrho^2+\delta^2}\right)<\frac{3\delta}{\varrho^4}$$

and thus from (6″):

$$\left|\xi\left(\frac{T_0^3}{\gamma}-1\right)\left(\frac{1}{(\sqrt{\varrho^2+\delta^2})^3}-\frac{1}{\varrho^3}\right)\right|<\frac{3\delta\cdot\varrho C\varrho^\lambda}{\varrho^4}=3\,C\,\delta\,\varrho^{\lambda-3}.$$

Moreover,

$$\begin{aligned}|T^2-T_0^2|&=\left|\frac{\varrho^2+\delta^2}{\varrho^2+(\delta-\zeta)^2}-\frac{\varrho^2}{\varrho^2+\zeta^2}\right|=\frac{T^2|\delta^2\zeta^2+2\varrho^2\delta\zeta|}{(\varrho^2+\delta^2)(\varrho^2+\zeta^2)}\\&=T^2\delta|\zeta|\frac{2\varrho^2+\delta|\zeta|}{(\varrho^2+\delta^2)(\varrho^2+\zeta^2)}<3\,T^2\frac{\delta|\zeta|}{\varrho^2}<\frac{27}{4}b\,\delta\,\varrho^{\lambda-1}.\end{aligned}$$

From this it follows, since the quantities $T^3-T_0^3$, $T^2-T_0^2$, and $T-T_0$ have the same sign, that the estimates

$$\begin{aligned}|T^3-T_0^3|&=|(T^2-T_0^2)(T+T_0)-T\,T_0(T-T_0)|\\&\le(T+T_0)\,|T^2-T_0^2|<\frac{81}{4}b\,\delta\,\varrho^{\lambda-1}.\end{aligned}$$

are valid.

Then

$$\left|\xi\frac{1}{(\sqrt{\varrho^2+\delta^2})^3}\cdot\frac{T^3-T_0^3}{\gamma}\right|<\varrho\cdot\frac{1}{\varrho^3}\cdot\frac{81b}{2}\,\delta\,\varrho^{\lambda-1}=\frac{81b}{2}\,\delta\,\varrho^{\lambda-3}.$$

Thus, the absolute value of integral (9) is not greater than

$$\begin{aligned}&4\pi c_1\delta\int\limits_{2\delta}^{R}\varrho^{\lambda-3}\varrho\,d\varrho\le4\pi c_1\delta\int\limits_{2\delta}^{d}\varrho^{\lambda-2}d\varrho\\&\qquad=\frac{4\pi c_1}{1-\lambda}\delta[(2\delta)^{\lambda-1}-d^{\lambda-1}]<\frac{4\pi c_1}{(1-\lambda)\,2^{1-\lambda}}\,\delta^\lambda=c_2\delta^\lambda.\end{aligned}$$

This completes the proof, since the integral (5) differs from its limit (8) by a quantity which is less than $c_3\delta^\lambda$ in absolute value. Thus finally:

$$\frac{\partial V_e}{\partial \xi} = \int\limits_{(S-\Sigma)} \mu \frac{\xi}{r_0^3}\, d\sigma + \int\limits_{(\Sigma)} (\mu - \mu_0) \frac{\xi}{r_0^3}\, d\sigma + \mu_0 \int\limits_{(\Sigma)} \left(\frac{T_0^3}{\gamma} - 1\right) \frac{\xi \gamma}{\varrho^3}\, d\sigma. \tag{10}$$

Replacing ξ by η, we obtain the corresponding formula for $\frac{\partial V_e}{\partial \eta}$ which, without writing it down, we shall denote by (10_1).

Now let (x,y,z) be some fixed coordinate system. Since

$$\left(\frac{\partial V}{\partial x}\right)_{M_1} = \left(\frac{\partial V}{\partial \xi}\right)_{M_1} \cos(\xi x) + \left(\frac{\partial V}{\partial \eta}\right)_{M_1} \cos(\eta x) + \left(\frac{\partial V}{\partial \zeta}\right)_{M_1} \cos(\zeta x),$$

as $\delta \to 0$ $\left(\frac{\partial V}{\partial x}\right)_{M_1}$ has a well-defined limit which is equal to

$$\frac{\partial V_e}{\partial x} = \frac{\partial V_e}{\partial \xi} \cos(\xi x) + \frac{\partial V_e}{\partial \eta} \cos(\eta x) + \frac{\partial V_e}{\partial \zeta} \cos(\zeta x). \tag{11}$$

It follows moreover from the estimates above that

$$\left| \left(\frac{\partial V}{\partial x}\right)_{M_1} - \frac{\partial V_e}{\partial x} \right| < c A \delta^\lambda, \tag{12}$$

since each summand on the right-hand side of (11) satisfies a similar inequality.

It has thus been proved that $\left(\frac{\partial V}{\partial x}\right)_{M_1}$ has a well-defined limit when the point M_1 approaches a point M_0 of (S) on the normal N_0. It must now be shown that $\frac{\partial V}{\partial x}$ converges to the same limit when M_1 approaches the point M_0 along an arbitrary path.

We assign to each point M of (S) that point M_δ of (D_e) which lies on the normal N to (S) at the point M at a distance of δ from M. The value of $\frac{\partial V}{\partial x}$ at the point M_δ is a continuous function of M, since two nearby points M' and M'' points of (S) correspond to two nearby points M'_δ and M''_δ.

According to (12), as $\delta \to 0 \left(\frac{\partial V}{\partial x}\right)_{M_\delta}$ converges uniformly to the limit $\frac{\partial V_e}{\partial x}$; $\frac{\partial V_e}{\partial x}$ is therefore a continuous function of the point M of (S). Let now M_0 be a point of (S) and M_1 a point of (D_e) at a distance δ from M_0; further let M_2 be the point of (S) nearest M_1. Then $r_{12} \leqq \delta$ and hence $r_{02} \leqq 2\delta$. If we now consider the inequality

$$\left| \left(\frac{\partial V}{\partial x}\right)_{M_1} - \left(\frac{\partial V_e}{\partial x}\right)_{M_0} \right| \leq \left| \left(\frac{\partial V}{\partial x}\right)_{M_1} - \left(\frac{\partial V_e}{\partial x}\right)_{M_2} \right| + \left| \left(\frac{\partial V_e}{\partial x}\right)_{M_2} - \left(\frac{\partial V_e}{\partial x}\right)_{M_0} \right|,$$

we see that as $\delta \to 0$ the left-hand side tends to the limit zero, since as $\delta \to 0$ the summands on the right-hand side tend to this limit. This proves our assertion.

It follows now that the function $\frac{\partial V}{\partial x}$ is continuous in a closed region and is therefore uniformly continuous and bounded there. From our estimates we may conclude that $\left|\frac{\partial V}{\partial x}\right|$ has an upper bound of the form cA.

2. We now wish to prove that the derivatives $\frac{\partial V_e}{\partial x}$, $\frac{\partial V_e}{\partial y}$, and $\frac{\partial V_e}{\partial z}$ are H-continuous on (S). Let M_0 be a point of the surface and (x,y,z) be a local coordinate system about M_0. Let M_1 be another point of the surface at a distance δ from M_0. Let us denote the local coordinate system about M_1 by (ξ,η,ζ). To prove the H-continuity of the limits of the first derivatives of the simple-layer potential, we show that the differences

$$\left(\frac{\partial V_e}{\partial x}\right)_{M_1} - \left(\frac{\partial V_e}{\partial x}\right)_{M_0}, \quad \left(\frac{\partial V_e}{\partial y}\right)_{M_1} - \left(\frac{\partial V_e}{\partial y}\right)_{M_0}, \quad \left(\frac{\partial V_e}{\partial z}\right)_{M_1} - \left(\frac{\partial V_e}{\partial z}\right)_{M_0} \tag{13}$$

are not greater than numbers of the form $aA\delta^{\lambda'}$ with $\delta = |M_0M_1|$ and $\lambda' < \lambda$. Since the proof for the second difference is similar to that for the first difference, we need consider only the first and third differences. We first turn to the third difference. We have:

$$\left(\frac{\partial V_e}{\partial z}\right)_{M_0} = \left(\frac{dV}{dn}\right)_{M_0} - 2\pi\mu_0$$

and from (11)

$$\left(\frac{\partial V_e}{\partial z}\right)_{M_1} = \left(\frac{\partial V_e}{\partial \xi}\right)_{M_1} \cos(\xi z) + \left(\frac{\partial V_e}{\partial \eta}\right)_{M_1} \cos(\eta z) + \left[\left(\frac{dV}{dn}\right)_{M_1} - 2\pi\mu_1\right] \cos(N_1 N_0).$$

Denoting by λ' an arbitrary number of the interval $0 < \lambda' < \lambda$, we now obtain the following inequality:

$$\begin{aligned}\left|\left(\frac{\partial V_e}{\partial z}\right)_{M_1} - \left(\frac{\partial V_e}{\partial z}\right)_{M_0}\right| &< cA(|\cos(\xi z)| + |\cos(\eta z)|) \\ &+ \left|\left(\frac{dV}{dn}\right)_{M_0} - \left(\frac{dV}{dn}\right)_{M_1} + 2\pi(\mu_1 - \mu_0)\right| \cos(N_1 N_0) \\ &+ \left|\left(\frac{dV}{dn}\right)_{M_0} - 2\pi\mu_0\right| [1 - \cos(N_1 N_0)] < c_1 A \delta^{\lambda'};\end{aligned}$$

for one has:

$$|\cos(\xi z)| = |\cos(\xi N_0) - \cos(\xi N_1)| \leq (N_1 N_0) < E\delta^\lambda;$$

$$|\cos(\eta z)| < E\delta^\lambda;$$

$$1 - \cos(N_1 N_0) = 2\sin^2\frac{(N_1 N_0)}{2} \leq (N_1 N_0) < E\delta^\lambda;$$

$$|\mu_1 - \mu_0| < A\delta^\lambda; \qquad \left|\left(\frac{dV}{dn}\right)_{M_0} - \left(\frac{dV}{dn}\right)_{M_1}\right| < cA\delta^{\lambda'}.$$

The last of the inequalities mentioned was proved in II, §5 under the hypothesis that μ be simply bounded. If μ is assumed to be H-continuous, then one can prove that this inequality holds with exponent λ. We shall make no use of this fact and therefore not pursue its proof further. The absolute value of the third difference in (13) can thus be estimated by a quantity of the form $c_1 A\delta^{\lambda'}$

We now proceed to the proof of the analogous assertion for the first of the differences (13).

3. To this end, we compute first of all the quantity $\left(\frac{\partial V}{\partial x}\right)_{M_1}$. The coordinates of the integration point M we shall denote by x,y,z or ξ,η,ζ according to whether it is a question of the local coordinate system about M_0 or that about M_1. The distance of the point M from M_1 we denote by r_1, and the distances M_0M and M_0M_1 by r_0 and δ respectively. Clearly,

$$\xi\cos(\xi x) + \eta\cos(\eta x) + \zeta\cos(\zeta x) = r_1\cos(r_1 x) = x - x_1, \tag{14}$$

where x_1 is the first coordinate of the point M_1.

Making use of the formula

$$\left(\frac{\partial V_e}{\partial \zeta}\right)_{M_1} = \int_{(S)} \mu\frac{\zeta}{r_1^3}\,d\sigma - 2\pi\mu_1$$

as well as formulas (10), (10_1), (11), and (14), we find:

$$\begin{aligned}\left(\frac{\partial V_e}{\partial x}\right)_{M_1} = &\int_{(S-\Sigma_1)} \mu\frac{x-x_1}{r_1^3}\,d\sigma + \int_{(\Sigma_1)}(\mu-\mu_1)\frac{x-x_1}{r_1^3}\,d\sigma \\ &+ \mu_1\int_{(\Sigma_1)}\left(\frac{T_1^3}{\gamma_1} - 1\right)\frac{(x-x_1)\gamma_1}{\varrho_1^3}\,d\sigma \\ &+ \cos(\zeta x)\left[-2\pi\mu_1 + \mu_1\int_{(\Sigma_1)}\frac{\zeta}{r_1^3}\,d\sigma - \mu_1\int_{(\Sigma_1)}\left(\frac{T_1^3}{\gamma_1} - 1\right)\frac{\zeta\gamma_1}{\varrho_1^3}\,d\sigma\right].\end{aligned} \tag{15}$$

Here $\gamma_1 = \cos(N_1 N)$; ϱ_1 denotes the projection of r_1 onto the (ξ,η) tangent plane at the point M_1; finally, (Σ_1) is the subregion of (S) whose projection onto the (ξ,η) plane is a circle of radius $R_1\left(\frac{d}{2} \leqq R_1 \leqq d\right)$ about M_1 which lies

inside the LYAPUNOV sphere about M_1. Since $|\zeta| < b\varrho_1^{1+\lambda}$ both integrals in square brackets are convergent; the absolute value of this bracket is no greater than a number of the form cA. Moreover,

$$|\cos(\zeta x)| = |\cos(N_1 x)| \\ = |\cos(N_1 x) - \cos(N_0 x)| \leq (N_1 N_0) < E\delta^\lambda .$$

Thus, the absolute value of the product of the square bracket and $\cos(\zeta x)$ is no greater than a number of the form $cA\delta^\lambda$; we shall subsequently no longer write down these summands in estimating the first of the differences of (13).

From (10) we obtain further:

$$\left(\frac{\partial V_e}{\partial x}\right)_{M_0} = \int\limits_{(S-\Sigma)} \mu \frac{x}{r_0^3} d\sigma + \int\limits_{(\Sigma)} (\mu - \mu_0) \frac{x}{r_0^3} d\sigma + \mu_0 \int\limits_{(\Sigma)} \left(\frac{T_0^3}{\gamma} - 1\right) \frac{x\gamma}{\varrho^3} d\sigma . \tag{16}$$

We first fix the subregion (Σ) by setting the radius R of its projection onto the (x,y) plane equal to $\frac{d}{2}$. In the following let us assume that $\delta < \frac{d}{4}$. The point M_1 then lies on (Σ). We shall choose the radius d of the LYAPUNOV sphere so small that parallels to the normals of a subregion of the surface contained in a LYAPUNOV sphere cut this subregion in at most one point. We now choose (Σ_1). Let (Σ') be the figure obtained by projecting (Σ) onto the (ξ,η) plane; it is clear that the point M_1 lies in the interior of (Σ'). Further let R_1 be the greatest and R_2 be the least distance of the point M_1 from points of the boundary line of (Σ'). We then choose (Σ_1) as that subregion of (S) whose projection onto the (ξ,η) plane is the circle with radius R_1 about M_1 (and which is contained in the interior of the LYAPUNOV sphere about M_1). It is clear that (Σ) is contained in (Σ_1).

We now estimate the area of the subregion of (Σ_1) lying outside (Σ). Since $\cos(N_1 N) > \frac{1}{2}$, this area is not greater than double the area of the projection of this region onto the (ξ,η) plane which is obviously contained in the annulus about M_1 with radii R_1 and R_2. Hence, the area of the surface $(\Sigma_1 - \Sigma)$ is not greater than $2\pi(R_1^2 - R_2^2)$. We shall show that this last quantity can be estimated by a number of the form $c\delta^\lambda$. For this we first of all consider the difference $T_1^2 - T_0^2$ for surface points M in the subregion of (Σ_1) outside the sphere with radius $r_0 = 2\delta$ about M_0. According to formula (7),

$$T_1^2 - T_0^2 = \frac{\varrho_1^2}{r_1^2} - \frac{\varrho^2}{r_0^2} = \sin^2(r_1 N_1) - \sin^2(r_0 N_0) = \cos^2(r_0 N_0) - \cos^2(r_1 N_1) \\ = [\cos(r_0 N_0) - \cos(r_1 N_1)]\,[\cos(r_0 N_0) + \cos(r_1 N_1)] ,$$

where, in order to have a specific case in mind, we shall assume that the segments MM_0 and MM_1 are oriented from M to M_0 and M_1 respectively.

One has further from inequality (17) of I, §1:

$$|\cos(r_0 N_0)| < E r_0^\lambda < 2^\lambda E \varrho^\lambda ;$$

from inequality (9) of II, §2 one has outside the sphere $r_0 = 2\delta$ the inequality $\frac{r_0}{2} < r_1 < \frac{3}{2} r_0$; thus,

$$\cos(r_1 N_1)| < E r_1^\lambda < \left(\frac{3}{2}\right)^\lambda E r_0^\lambda < 3^\lambda E \varrho^\lambda .$$

Further,

$$| \cos(r_0 N_0) - \cos(r_1 N_1) | \leq | \cos(r_0 N_0) - \cos(r_0 N_1) |$$
$$+ |\cos(r_0 N_1) - \cos(r_1 N_1)| \leq (N_0 N_1) + (r_0 r_1) < E\delta^\lambda + (r_0 r_1).$$

From the triangle $M_0 M_1 M$ there follows the relation

$$\frac{\sin(r_0 r_1)}{\delta} = \frac{\sin(r_1 r_{01})}{r_0};$$

hence,

$$\sin(r_0 r_1) \leq \frac{\delta}{r_0} \leq \frac{\delta}{\varrho}.$$

Since $M_0 M_1$ is the smallest side of the triangle $M_0 M_1 M$, the sides MM_0 and MM_1 form an acute angle; thus,

$$(r_0 r_1) \leq \frac{\pi}{2} \sin(r_0 r_1) \leq \frac{\pi}{2} \cdot \frac{\delta}{\varrho}.$$

From all these estimates it now follows:

$$|T_1^2 - T_0^2| < \left[E\delta^\lambda + \frac{\pi}{2}\frac{\delta}{\varrho}\right] E\varrho^\lambda (3^\lambda + 2^\lambda) = c_1 \delta^\lambda \varrho^\lambda + c_2 \delta \varrho^{\lambda-1},$$

From $\frac{1}{2} < \frac{\varrho}{r_0} \leqq 1$ and $\frac{1}{2} < \frac{\varrho_1}{r_1} \leqq 1$ one has $\frac{1}{2} < T_0 \leqq 1$ and $\frac{1}{2} < T_0 \leqq 1$. It now follows that $T_1 + T_0 > 1$; therefore, from the preceding inequality

$$|T_1 - T_0| \leq |T_1^2 - T_0^2| < c_1 \delta^\lambda \varrho^\lambda + c_2 \delta \varrho^{\lambda-1}. \tag{17}$$

Since $T_0 + T_1 \leqq 2$,

$$|T_1^3 - T_0^3| \leq |(T_1^2 - T_0^2)(T_1 + T_0)| < 2 c_1 \delta^\lambda \varrho^\lambda + 2 c_2 \delta \varrho^{\lambda-1}. \tag{18}$$

Let M be a point of the boundary line of (Σ). Since $\delta < \frac{d}{4}$ it lies outside the sphere of radius $r_0 = 2\delta$. It thus satisfies the conditions of inequality (17). Thus, for this point

$$\varrho_1 = r_1 \sin(r_1 N_1) = r_1 T_1 \leq (r_0 + \delta) T_1 = r_0 T_0 + r_0(T_1 - T_0) + \delta T_1$$
$$< \frac{d}{2} + d\left[c_1 \delta^\lambda \left(\frac{d}{2}\right)^\lambda + c_2 \delta \left(\frac{d}{2}\right)^{\lambda-1}\right] + \delta = \frac{d}{2} + c_3 \delta^\lambda + c_4 \delta < \frac{d}{2} + c_5 \delta^\lambda;$$

hence,

$$R_1 = \operatorname{Max} \varrho_1 < \frac{d}{2} + c_5 \delta^\lambda .$$

Similarly we find the estimate

$$\varrho_1 = r_1 \sin(r_1 N_1) = r_1 T_1 \geq (r_0 - \delta) T_1 > \frac{d}{2} - c_3 \delta^\lambda - c_4 \delta > \frac{d}{2} - c_5 \delta^\lambda$$

and therefore

$$R_2 = \operatorname{Min} \varrho_1 > \frac{d}{2} - c_5 \delta^\lambda .$$

Thus,

$$0 \leq R_1^2 - R_2^2 < 2\, d\, c_6\, \delta^\lambda ,$$

which means that the area of the surface $(\Sigma_1 - \Sigma)$ is indeed bounded by a number of the form $c\delta^\lambda$.

Since on $(\Sigma_1 - \Sigma)$ r_1 is greater than $\frac{d}{4}$ and ϱ_1 is there greater than $\frac{d}{8}$, the functions

$$\frac{x - x_1}{r_1^3}, \qquad \left(\frac{T_1^3}{\gamma_1} - 1\right) \frac{(x - x_1)\,\gamma_1}{\varrho_1^3}$$

are bounded on $(\Sigma_1 - \Sigma)$; thus, if one replaces (Σ_1) by (Σ) in the first three integrals in (15) these change by an amount which is less in absolute value than a number of the form $cA\delta^\lambda$.

It thus remains to study the difference between expression (16) and the sum

$$\int\limits_{(S-\Sigma)} \mu \frac{x - x_1}{r_1^3} d\sigma + \int\limits_{(\Sigma)} (\mu - \mu_1) \frac{x - x_1}{r_1^3} d\sigma + \mu_1 \int\limits_{(\Sigma)} \left(\frac{T_1^3}{\gamma_1} - 1\right) \frac{(x - x_1)\,\gamma_1}{\varrho_1^3} d\sigma . \tag{19}$$

The absolute value of the difference of the first integrals in (16) and (19) is obviously no greater than a number of the form $cA\delta$.

From $|\mu_1 - \mu_0| < A\delta^\lambda$ and the boundedness of the third integral in (16) the difference of the third summands in (16) and (19) differs from the product of μ_1 and the difference of the third integrals by a quantity which is bounded in absolute value by a number of the form $cA\delta^\lambda$. It will further be shown that the difference of the third integrals is no greater in absolute value than a number of the form $c\delta^{\lambda'}$ with $\lambda' < \lambda$. From this it then follows that the absolute value of the difference of the third summands in (16) and (19) is no greater than a number of the form $cA\delta^{\lambda'}$.

Let (σ) be that subregion of (Σ) whose projection onto the (x,y) plane is the circle of radius 2δ about M_0. It is then easy to see that the integrals over (σ) of the integrands in the second and third summands of (16) and (19) are less in absolute value than numbers of the form $cA\delta^\lambda$. It thus remains to estimate the difference of the corresponding integrals over $(\Sigma - \sigma)$. We first of all consider the difference of the second integrals. Since on $(\Sigma - \sigma)$

$$\frac{1}{r_1} < \frac{2}{r_0} \leq \frac{2}{\varrho} ,$$

it follows that there

$$\left|(\mu-\mu_1)\frac{x-x_1}{r_1^3}-(\mu-\mu_0)\frac{x}{r_0^3}\right|$$
$$=\left|(\mu-\mu_0)x\left(\frac{1}{r_1^3}-\frac{1}{r_0^3}\right)-(\mu-\mu_0)\frac{x_1}{r_1^3}+(\mu_0-\mu_1)\frac{x-x_1}{r_1^3}\right.$$
$$< A\,r_0^\lambda\cdot\varrho\frac{c\,\delta}{\varrho^4}+A\,r_0^\lambda\cdot\frac{8\,\delta}{\varrho^3}+A\,\delta^\lambda\frac{4}{\varrho^2}<(c_1\,\delta\,\varrho^{\lambda-3}+4\,\delta^\lambda\,\varrho^{-2})A;$$

hence, the absolute value of the difference of the second integrals over $(\Sigma-\sigma)$ is no greater than

$$4\pi A\left[c_1\delta\int_{2\delta}^{\frac{d}{2}}\varrho^{\lambda-3}\varrho\,d\varrho+4\,\delta^\lambda\int_{2\delta}^{\frac{d}{2}}\frac{\varrho\,d\varrho}{\varrho^2}\right]$$
$$< c_2 A\,\delta^\lambda+16\pi A\,\delta^\lambda\ln\frac{d}{\delta}<c_3 A\,\delta^{\lambda'}\qquad(0<\lambda'<\lambda)$$

It remains to investigate the difference of the third integrals in (16) and (19) over $(\Sigma-\sigma)$. Note first of all that on $(\Sigma-\sigma)$ the following inequalities hold:

$$\varrho_1=r_1T_1>\frac{1}{2}r_1>\frac{1}{2}\frac{r_0}{2}=\frac{r_0}{4}\geq\frac{\varrho}{4};$$
$$\varrho_1=r_1T_1\leq r_1<\frac{3}{2}r_0\leq\frac{3}{2}\cdot 2\,\varrho=3\,\varrho.$$

from these follows:

$$\frac{\varrho}{4}<\varrho_1<3\,\varrho. \tag{20}$$

From inequality (17) we then have:

$$|\varrho_1-\varrho|=|r_1T_1-r_0T_0|=|(r_1-r_0)T_1+r_0(T_1-T_0)$$
$$<\delta+2\varrho(c_1\delta^\lambda\varrho^\lambda+c_2\delta\,\varrho^{\lambda-1})$$
$$=\delta(1+2\,c_2\,\varrho^\lambda)+\delta^\lambda 2\,c_1\varrho^\lambda\cdot\varrho<c_3\delta+c_4\,\varrho\,\delta^\lambda$$

and hence from (20):

$$\left|\frac{1}{\varrho_1^3}-\frac{1}{\varrho^3}\right|=\frac{|\varrho-\varrho_1|}{\varrho\,\varrho_1}\left(\frac{1}{\varrho_1^2}+\frac{1}{\varrho_1\varrho}+\frac{1}{\varrho^2}\right)<\frac{84\,(c_3\delta+c_4\varrho\,\delta^\lambda)}{\varrho^4} \tag{21}$$

Further, since $T_0+1>\dfrac{3}{2}>1$:

$$|T_0-1|=\frac{|T_0^2-1|}{T_0+1}<|T_0^2-1|$$
$$=\cos^2(r_0N_0)<E^2r_0^{2\lambda}<(E^2r_0^\lambda)(2\,\varrho)^\lambda<c_5\,\varrho^\lambda.$$

Since $T_0^2+T_0+1\leqq 3$,

$$|T_0^3-1|<3\,c_5\,\varrho^\lambda.$$

Therefore, since $1 - \gamma < E(2\varrho)^{\lambda}$,

$$|T_0^3 - \gamma| \leq |T_0^3 - 1| + |1 - \gamma| < c_5' \varrho^{\lambda} \tag{22}$$

and similarly

$$|T_1^3 - \gamma_1| < c_5' \varrho_1^{\lambda} < c_6 \varrho^{\lambda}; \tag{22'}$$

moreover,

$$|\gamma - \gamma_1| = |\cos(NN_0) - \cos(NN_1)| \leq (N_0 N_1) < E\delta^{\lambda}. \tag{22''}$$

Using (18), (20)–(22″), and the inequalities $|x| \leqq \varrho$ and $|x_1| \leqq \delta$, we obtain on $(\Sigma - \sigma)$:

$$\begin{aligned}
&\left|\frac{T_1^3 - \gamma_1}{\varrho_1^3}(x - x_1) - \frac{T_0^3 - \gamma}{\varrho^3} x\right| \\
&\leq \left|x_1 \frac{T_1^3 - \gamma_1}{\varrho_1^3}\right| + \left|x \frac{(T_1^3 - T_0^3) + (\gamma - \gamma_1)}{\varrho_1^3}\right| + \left|x\left(\frac{1}{\varrho_1^3} - \frac{1}{\varrho^3}\right)(T_0^3 - \gamma)\right| \\
&< \delta \frac{64 c_6 \varrho^{\lambda}}{\varrho^3} + \varrho \frac{64[(2c_1\delta^{\lambda}\varrho^{\lambda} + 2c_2\delta\varrho^{\lambda-1}) + E\delta^{\lambda}]}{\varrho^3} + \varrho \frac{84(c_3\delta + c_4\varrho\delta^{\lambda})}{\varrho^4} \cdot c_5' \varrho^{\lambda} \\
&= \delta\varrho^{\lambda-3}(64c_6 + 128c_2 + 84c_3c_5') + \delta^{\lambda}\varrho^{-2}(128c_1\varrho^{\lambda} + 64E + 84c_4c_5'\varrho^{\lambda}) \\
&< c_7\delta\varrho^{\lambda-3} + c_8\delta^{\lambda}\varrho^{-2}.
\end{aligned}$$

From this it now follows that the integral over $(\Sigma - \sigma)$ of the left-hand side of the last inequality is no greater than

$$4\pi c_7 \delta \int_{2\delta}^{\frac{d}{2}} \varrho^{\lambda-3} \varrho\, d\varrho + 4\pi c_8 \delta^{\lambda} \int_{2\delta}^{\frac{d}{2}} \varrho^{-2} \varrho\, d\varrho$$

$$< \frac{4\pi c_7}{1-\lambda}\delta^{\lambda} + 4\pi c_8 d^{\lambda} \ln\frac{d}{4\delta} < c_9\delta^{\lambda} + c_9'\delta^{\lambda'} < c_{10}\delta^{\lambda'}.$$

This completes the proof of the theorem.

§II. The Theorems of LYAPUNOV on the Normal Derivative of the Potential of the Double Layer

1. **Theorem 1.** *Suppose one of the following two conditions is satisfied*:

1. *μ is continuous on the surface (S), and $(N_1N_2) < Er_{12}$ (r_{12} is the distance between the two points M_1 and M_2 of the surfaces (S); N_1 and N_2 are the normals to (S) at these points).*

2. $$|\mu_{M_1} - \mu_{M_2}| < A r_{12}, \qquad (N_1N_2) < E r_{12}^{\lambda}.$$

If then the potential

$$W = \int_{(S)} \mu \frac{\cos(rN)}{r^2} d\sigma \text{ [1]} \tag{1}$$

possesses one of the derivatives

$$\frac{dW_i}{dn}, \qquad \frac{dW_e}{dn}, \tag{2}$$

then it also has the other, and

$$\frac{dW_i}{dn} = \frac{dW_e}{dn}.$$

Proof. If μ_0 denotes the value of μ at the surface point M_0, then

$$W = \int_{(S)} (\mu - \mu_0) \frac{\cos(rN)}{r^2} d\sigma + \mu_0 \int_{(S)} \frac{\cos(rN)}{r^2} d\sigma$$
$$= \int_{(S)} (\mu - \mu_0) \frac{\cos(rN)}{r^2} d\sigma + \mu_0 \eta;$$

here $\eta = 0$ or $\eta = 4\pi$ according to whether the function value of W is taken at a point M belonging to (D_e) or to (D_i).

From the last equation it follows: The derivatives $\frac{dW_i}{dn}$ or $\frac{dW_e}{dn}$—we denote them both by $\frac{dW}{dn}$—exist only if the corresponding integrals

$$\frac{d}{dn} \int_{(S)} (\mu - \mu_0) \frac{\cos(rN)}{r^2} d\sigma$$

exists, and then

$$\frac{dW}{dn} = \frac{d}{dn} \int_{(S)} (\mu - \mu_0) \frac{\cos(rN)}{r^2} d\sigma.$$

We choose the normal N_0 to (S) at M_0 as the z axis of a local coordinate system (x,y,z). Let (Σ) be the subregion of (S) cut out by the circular cylinder of radius d with N_0 as axis.

We choose the radius d such that the inequality

$$a d^\lambda < \frac{1}{4}$$

is satisfied and that (Σ) lies in the interior of a Lyapunov sphere about M_0. If ξ,η,ζ are the coordinates of a point M_1 of (S) and ϱ is the projection of the segment M_0M_1 onto the (x,y) plane, then for points of (Σ)

$$|\zeta| < a \varrho^{1+\lambda} \qquad (\lambda \leq 1).$$

We have:

$$\frac{dW}{dn} = \frac{d}{dn}\int\limits_{(\Sigma)} (\mu - \mu_0)\frac{\cos(rN)}{r^2}\,d\sigma + \frac{d}{dn}\int\limits_{(S-\Sigma)} (\mu - \mu_0)\frac{\cos(rN)}{r^2}\,d\sigma\,.$$

Since in the integral over $(S - \Sigma)$ the outer and inner normal derivative at M_0 exist and are equal, we need only concern ourselves with the outer and inner normal derivative of the integral over (Σ). In the coordinate system we have chosen

$$\frac{d}{dn}\int\limits_{(\Sigma)} (\mu - \mu_0)\frac{\cos(rN)}{r^2}\,d\sigma = \frac{\partial}{\partial z}\int\limits_{(\Sigma)} (\mu - \mu_0)\frac{\cos(rN)}{r^2}\,d\sigma\,.$$

Now for a point $M(0,0,z)$ on N_0 different from M_0

$$\frac{\partial}{\partial z}\int\limits_{(\Sigma)} (\mu - \mu_0)\frac{\cos(rN)}{r^2}\,d\sigma = \int\limits_{(\Sigma)} (\mu - \mu_0)\left\{\frac{1}{r^2}\frac{\partial\cos(rN)}{\partial z} + \frac{2\cos(rN)\cos(rz)}{r^3}\right\}d\sigma\,.$$

If one takes into account the relation

$$\cos(rN) = \frac{\xi}{r}\cos(Nx) + \frac{\eta}{r}\cos(Ny) + \frac{\zeta - z}{r}\cos(Nz)$$

with

$$r^2 = \xi^2 + \eta^2 + (\zeta - z)^2,$$

then it follows that

$$\begin{aligned}\frac{\partial\cos(rN)}{\partial z} &= \frac{\partial\frac{1}{r}}{\partial z}\,\xi\cos(Nx) + \frac{\partial\frac{1}{r}}{\partial z}\,\eta\cos(Ny)\\ &\quad + \frac{\partial\frac{1}{r}}{\partial z}(\zeta - z)\cos(Nz) - \frac{1}{r}\cos(Nz)\\ &= \frac{\cos(rz)}{r}\left\{\frac{\xi}{r}\cos(Nx) + \frac{\eta}{r}\cos(Ny) + \frac{\zeta - z}{r}\cos(Nz)\right\} - \frac{1}{r}\cos(Nz)\\ &= \frac{\cos(rz)\cos(rN)}{r} - \frac{1}{r}\cos(Nz)\,;\end{aligned}$$

whence

$$\frac{\partial}{\partial z}\int\limits_{(\Sigma)} (\mu - \mu_0)\frac{\cos(rN)}{r^2}\,d\sigma = \int\limits_{(\Sigma)} (\mu - \mu_0)\frac{3\cos(rz)\cos(rN) - \cos(Nz)}{r^3}\,d\sigma\,. \tag{3}$$

We now choose a number $R < d$ and consider the circular cylinder of radius R and axis coincident with N_0; let (σ_0) be the subregion of (Σ) cut out by this circular cylinder, and let $\Omega(z,R)$ be the integral with the same integrand as on the right-hand side of (3) but extended over the surface $(\Sigma - \sigma_0)$. The integral

(3) thus becomes $\Omega(z,0)$. It is then the same integral as on the right-hand side of (3), but extended over the surface (σ_0) it is equal to $\Omega(z,0) - \Omega(z,R)$. Introducing polar coordinates in the (x,y) plane with M_0 as origin, we obtain:

$$\Omega(z,0) - \Omega(z,R) = \int_0^{2\pi} d\varphi \int_0^R \frac{\mu - \mu_0}{r^3} \left\{ \frac{3\cos(rz)\cos(rN)}{\cos(Nz)} - 1 \right\} \varrho \, d\varrho \, .$$

We now consider the function under the integral sign. We have:

$$\frac{1}{r^3}\left\{\frac{3\cos(rz)\cos(rN)}{\cos(Nz)} - 1\right\} = \frac{1}{r^4}\left\{\frac{3\cos(rz)\, r\cos(rN)}{\cos(Nz)} - r\right\}.$$

If M lies in the interior of (D_i) (Fig. 34), then

$$r\cos(rN) = r_0\cos(r_0 N) + |z|\cos(Nz) = r_0\cos(r_0 N) - z\cos(Nz);$$

if M lies in the interior of (D_e), then

$$r\cos(rN) = r_0\cos(r_0 N) - |z|\cos(Nz) = r_0\cos(r_0 N) - z\cos(Nz).$$

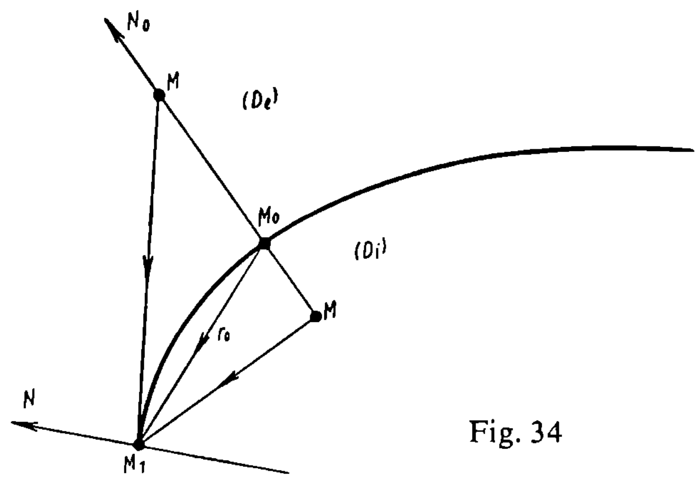

Fig. 34

From this it follows in both cases:

$$\begin{aligned}
\frac{3\cos(rz)\, r\cos(rN)}{\cos(Nz)} - r &= \frac{3\cos(rz)\, r_0\cos(r_0 N)}{\cos(Nz)} - 3\cos(rz)z - r \\
&= \frac{3(\zeta - z)\, r_0\cos(r_0 N)}{r\cos(Nz)} - \frac{3(\zeta - z)z + r^2}{r} \\
&= \frac{3(\zeta - z)\, r_0\cos(r_0 N)}{r\cos(Nz)} - \frac{3(\zeta - z)z + \varrho^2 + (\zeta - z)^2}{r} \\
&= \frac{1}{r}\left\{\frac{3(\zeta - z)\, r_0\cos(r_0 N)}{\cos(Nz)} - \zeta z - \zeta^2 - \varrho^2 + 2z^2\right\},
\end{aligned}$$

for

$$\xi^2 + \eta^2 = \varrho^2 .$$

We thus obtain:

$$\Omega(z, 0) - \Omega(z, R) \tag{4}$$
$$= \int_0^{2\pi} d\varphi \int_0^R (\mu - \mu_0) \left\{ -\frac{\varrho^2 - 2z^2}{r^5} + \frac{1}{r^5} \left[\frac{3(\zeta - z) r_0 \cos(r_0 N)}{\cos(N z)} - \zeta z - \zeta^2 \right] \right\} \varrho \, d\varrho .$$

We now put

$$r_1 = \sqrt{\varrho^2 + z^2};$$

then

$$r^2 - r_1^2 = \varrho^2 + (\zeta - z)^2 - \varrho^2 - z^2 = \zeta^2 - 2\zeta z$$

and

$$|r^2 - r_1^2| < a^2 \varrho^2 \varrho^{2\lambda} + 2a\varrho\varrho^\lambda |z|,$$
$$r^2 = r_1^2 + \Theta(a^2 \varrho^2 \varrho^{2\lambda} + 2a\varrho\varrho^\lambda |z|) \quad \text{with} \quad |\Theta| < 1,$$
$$r = r_1 \sqrt{1 + \frac{\Theta(a^2\varrho^2\varrho^{2\lambda} + 2a\varrho\varrho^\lambda |z|)}{r_1^2}}$$
$$= r_1 \sqrt{1 + \Theta_1 \left(a^2 \varrho^{2\lambda} + 2a\varrho^\lambda \frac{|z|}{r_1} \right)} \quad \text{with} \quad |\Theta_1| < 1.$$

We note that the second term under the last radical on the basis of the properties of the subregion (Σ) of (S) we have chosen is less than

$$\frac{1}{16} + 2 \cdot \frac{1}{4} < \frac{3}{4};$$

hence,

$$\frac{1}{r^5} = \frac{1}{r_1^5} + g \frac{a^2 \varrho^{2\lambda} + 2a\varrho^\lambda \frac{|z|}{r_1}}{r_1^5}, \qquad \frac{1}{r^5} = \frac{h}{r_1^5},$$

where g and h are bounded functions.

We can now write equation (4) in the following form:

$$\Omega(z, 0) - \Omega(z, R) = -\int_0^{2\pi} d\varphi \int_0^R (\mu - \mu_0) \frac{\varrho^2 - 2z^2}{r_1^5} \varrho \, d\varrho$$
$$+ \int_0^{2\pi} d\varphi \int_0^R (\mu - \mu_0) \frac{Q}{r_1^5} \varrho \, d\varrho \tag{5}$$

with

$$Q = -g\left(a^2 \varrho^{2\lambda} + 2a \frac{\varrho^\lambda |z|}{r_1}\right)(\varrho^2 - 2z^2)$$
$$+ h \left\{ \frac{3(\zeta - z) r_0 \cos(r_0 N)}{\cos(N z)} - \zeta z - \zeta^2 \right\}.$$

We wish to find an upper bound for Q. If we estimate each term of Q and drop the constant factors, we obtain the following upper bounds:

$$\varrho^2 \varrho^{2\lambda}, \quad \varrho\varrho^\lambda |z|, \quad \varrho^{2\lambda} |z|^2, \quad \varrho^\lambda |z|^2, \quad \varrho^2 \varrho^{2\lambda}, \quad \varrho\varrho^\lambda |z|, \quad \varrho\varrho^\lambda |z|, \quad \varrho^2 \varrho^{2\lambda}.$$

Indeed,

$$|\cos(r_0 N)| < b\varrho^\lambda, \quad r_0 < 2\varrho \quad (\varrho = r_0 \cos\alpha,\ \cos\alpha > \tfrac{1}{2}).$$

From this there follows the inequality

$$|Q| < A\varrho^2\varrho^{2\lambda} + B\varrho\varrho^\lambda |z| + C\varrho^\lambda |z|^2,$$

where A, B, and C are constants.

We now turn to the hypotheses of our theorem. One can choose R so small that for points of (σ_0) the following inequalities hold:

$$\left.\begin{array}{lll} |\mu-\mu_0| < \varepsilon, & (NN_0) < b\varrho, \quad \lambda = 1 & \text{in case } 1^\circ, \\ |\mu-\mu_0| < a\varrho, & (NN_0) < b\varrho^\lambda < \varepsilon & \text{in case } 2^\circ. \end{array}\right\} \tag{6}$$

In both these cases we have:

$$|\mu-\mu_0|\,|Q| < \varepsilon\{A\varrho^3\varrho^\lambda + B\varrho^2|z| + C\varrho|z|^2\}.$$

In order to estimate the second term in (5), the integrals

$$A\int_0^R \frac{\varrho^4\varrho^\lambda\, d\varrho}{r_1^5}, \qquad B|z|\int_0^R \frac{\varrho^3\, d\varrho}{r_1^5}, \qquad C|z|^2\int_0^R \frac{\varrho^2\, d\varrho}{r_1^5} \tag{7}$$

must be estimated. We find:

$$\int_0^R \frac{\varrho^4\varrho^\lambda\, d\varrho}{r_1^5} < \int_0^R \frac{\varrho^\lambda\, d\varrho}{\varrho} = \frac{1}{\lambda} R^\lambda,$$

$$|z|\int_0^R \frac{\varrho^3\, d\varrho}{r_1^5} < |z|\int_0^R \frac{\varrho\, d\varrho}{\sqrt{(\varrho^2+z^2)^3}} = |z|\left\{\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}\right\} < 1,$$

$$|z|^2\int_0^R \frac{\varrho^2\, d\varrho}{r_1^5} < |z|^2\int_0^R \frac{d\varrho}{\sqrt{(\varrho^2+z^2)^3}} = \frac{\varrho}{\sqrt{\varrho^2+z^2}}\Big|_0^R = \frac{R}{\sqrt{R^2+z^2}} < 1.$$

The three integrals in (7) are thus bounded, and

$$\Omega(z,0) - \Omega(z,R) = -\int_0^{2\pi} d\varphi \int_0^R (\mu-\mu_0)\frac{\varrho^2-2z^2}{r_1^5}\varrho\, d\varrho + \varepsilon K_1, \tag{8}$$

where K_1 is a bounded function.

If in (5) we replace z by $-z$, then we obtain in the same manner:

$$\Omega(-z,0) - \Omega(-z,R) = -\int_0^{2\pi} d\varphi \int_0^R (\mu-\mu_0)\frac{\varrho^2-2z^2}{r_1^5}\varrho\, d\varrho + \varepsilon K_2.$$

We thus find:

$$\Omega(z,0) - \Omega(-z,0) = \Omega(z,R) - \Omega(-z,R) + \varepsilon(K_1 - K_2).$$

Having fixed R, we choose z such that

$$|\Omega(z,R)-\Omega(-z,R)|<\varepsilon\quad;$$

this is possible, since $\Omega(z,R)$ has a limit as $z\to 0$. From this it follows:

$$|\Omega(z,0)-\Omega(-z,0)|<\varepsilon L\quad\text{mit}\quad L=1+|K_1|+|K_2|.$$

The last inequality implies the following: If one of the quantities

$$\Omega(z,0),\quad \Omega(-z,0)$$

has a limit as $z\to 0$, then the other converges to the same limit. This is what was required to prove.

Remark. In case 2 the inequality $(N_1N_2)<E\varrho^\lambda$ holds, so one can even assume that

$$|\mu-\mu_0|<a\varrho|\ln\varrho|.$$

Indeed, if

$$b\varrho^\lambda=b\varrho^{\lambda-\eta}\varrho^\eta\qquad(0<\eta<\lambda)$$

one can by choice of a sufficiently small R achieve that $b\,\varrho^{\lambda-n}$ is less than ε. Since $\varrho^n\ln\varrho$ remains bounded, the inequality obtained for $|\mu-\mu_0||Q|$ retains its form.

2. Let the point (x_0,y_0,z_0) lie on the boundary of (σ_0). We put

$$x_0=R\cos\varphi,\quad y_0=R\sin\varphi.$$

Theorem II. *If the density μ of the double-layer potential* (1)

$$W=\int_{(S)}\mu\frac{\cos(rN)}{r^2}\,d\sigma$$

satisfies one of the conditions of Theorem I and in addition a LYAPUNOV *condition*

$$\left|\int_0^{2\pi}[\mu(\varrho,\varphi)-\mu_0]\,d\varphi\right|<a\varrho^{1+\nu}\qquad(\nu>0),\tag{9}$$

then the potential W has a normal derivative at the point M_0.

Remark. One can generalize condition 2 and put:

$$|\mu_{M_1}-\mu_{M_2}|<Ar_{12}|\ln r_{12}|,\qquad\text{if}\quad(N_1N_2)<Er_{12}^\lambda.$$

Proof. Inequality (9) makes it possible to estimate the first integral in formula (5)

$$\Omega(z,0)-\Omega(z,R)$$
$$=-\int_0^{2\pi}d\varphi\int_0^R(\mu-\mu_0)\frac{\varrho^2-2z^2}{r_1^5}\varrho\,d\varrho+\int_0^{2\pi}d\varphi\int_0^R(\mu-\mu_0)\frac{Q}{r_1^5}\varrho\,d\varrho\;.$$

Taking into account the inequalities

$$\left|\frac{\varrho^4}{r_1^4}\right| < 1, \qquad \left|\frac{z^2\varrho^2}{r_1^4}\right| < 1$$

we find:

$$\left|\int_0^{2\pi} d\varphi \int_0^R (\mu - \mu_0)\frac{\varrho^2 - 2z^2}{r_1^5}\varrho\, d\varrho\right|$$

$$= \left|\int_0^R \frac{\varrho^2 - 2z^2}{r_1^5}\varrho\left(\int_0^{2\pi}(\mu - \mu_0)\, d\varphi\right) d\varrho\right|$$

$$\leq \int_0^R \left|\frac{\varrho^2 - 2z^2}{r_1^5}\right| \varrho \left|\int_0^{2\pi}(\mu - \mu_0)\, d\varphi\right| d\varrho$$

$$< a\int_0^R \frac{|\varrho^2 - 2z^2|\,\varrho^2\varrho^\nu}{r_1^5}\, d\varrho < 3a\int_0^R \frac{\varrho^\nu}{\varrho}\, d\varrho = \frac{3a}{\nu}R^\nu.$$

Recalling the estimate found for $|Q|$ and denoting by ε an upper bound for $|\mu - \mu_0|$ and $b\varrho^\lambda$ or $b\varrho^{\lambda - n}$ on (σ_0), we arrive at the following estimate:

$$|\Omega(z, 0) - \Omega(z, R)| < \frac{3a}{\nu}R^\nu + \varepsilon K. \tag{10}$$

We consider the circular cylinder with axis N_0 and radius $R_1 < R$. Let (σ_1) be the subregion which it cuts out from (Σ); $\Omega(z, R_1)$ then shall denote the integral with the same integrand as on the right-hand side of (3) extended over the surface $(\Sigma - \sigma_1)$. The integral over the surface between the two cylinders is equal to

$$\Omega(z, R_1) - \Omega(z, R)$$
$$= -\int_0^{2\pi} d\varphi \int_{R_1}^R \frac{(\varrho^2 - 2z^2)\varrho}{r_1^5}(\mu - \mu_0)\, d\varrho + \int_0^{2\pi} d\varphi \int_{R_1}^R (\mu - \mu_0)\frac{Q}{r_1^5}\varrho\, d\varrho.$$

For $z = 0$ the last equation gives:

$$\Omega(0, R_1) - \Omega(0, R) = -\int_0^{2\pi} d\varphi \int_{R_1}^R \frac{(\mu - \mu_0)\, d\varrho}{\varrho^2} + \int_0^{2\pi} d\varphi \int_{R_1}^R \frac{(\mu - \mu_0)\,\Theta A\varrho^{2\lambda}}{\varrho^2}\, d\varrho;$$

indeed, for $z = 0$

$$r_1 = \varrho, \quad Q = \Theta A\varrho^2\varrho^{2\lambda} \quad (|\Theta| < 1).$$

In the second term

$$|\mu - \mu_0|\,|\Theta|\,A\varrho^{2\lambda} < \varepsilon A\varrho\varrho^\lambda;$$

from this there follows the estimate:

$$\left|\int_0^{2\pi} d\varphi \int_{R_1}^{R} \frac{(\mu-\mu_0)\,\Theta A \varrho^{2\lambda}}{\varrho^2}\, d\varrho\right| < 2\pi\varepsilon A \int_{R_1}^{R} \frac{\varrho^\lambda}{\varrho}\, d\varrho = \frac{2\pi\varepsilon A}{\lambda}(R^\lambda - R_1^\lambda) < c\,\varepsilon R^\lambda .$$

Using the LYAPUNOV condition (9), one obtains for the first term:

$$\left|\int_0^{2\pi} d\varphi \int_{R_1}^{R} \frac{(\mu-\mu_0)\,d\varrho}{\varrho^2}\right| = \left|\int_{R_1}^{R} \frac{d\varrho}{\varrho^2}\int_0^{2\pi}(\mu-\mu_0)\,d\varphi\right|$$

$$\leq \int_{R_1}^{R} \frac{d\varrho}{\varrho^2}\left|\int_0^{2\pi}(\mu-\mu_0)\,d\varphi\right| < a\int_{R_1}^{R}\frac{\varrho^\nu\,d\varrho}{\varrho} = \frac{a}{\nu}(R^\nu - R_1^\nu) < \frac{a}{\nu}R^\nu .$$

There now follows the inequality

$$|\Omega(0,R_1) - \Omega(0,R)| < a R^\nu + \varepsilon b R^\lambda , \tag{11}$$

on the right side of which R_1 no longer occurs.

Since the right-hand side of (11) becomes arbitrarily small with R, we conclude that as $R \to 0$ $\Omega(0,R)$ tends to a limit which we denote by L:

$$\lim_{R\to 0}\Omega(0, R) = L .$$

As R_1 goes to zero, we obtain from (11):

$$|\Omega(0,R) - L| < a R^\nu + \varepsilon b R^\lambda . \tag{12}$$

We now investigate the difference

$$\Omega(z, 0) - L$$

between the value of the expression

$$\frac{d}{dn}\int_{(\Sigma)} (\mu-\mu_0)\frac{\cos(rN)}{r^2}\,d\sigma$$

at the point $M(0,0,z)$ and the quantity L just found. From

$$\Omega(z,0) - L = [\Omega(z,0) - \Omega(z,R)] + [\Omega(0,R) - L] + [\Omega(z,R) - \Omega(0,R)]$$

it follows using inequalities (10) and (12) that

$$|\Omega(z,0) - L| < a R^\nu + \varepsilon K + \varepsilon b R^\lambda + |\Omega(z,R) - \Omega(0,R)| .$$

The quantity R was chosen for arbitrarily given ε in such a manner that inequalities (6) are satisfied. We shall moreover assume that

$$a R^\nu < \varepsilon .$$

We then obtain:

$$|\Omega(z,0) - L| < \varepsilon B + |\Omega(z,R) - \Omega(0,R)| .$$

Without changing the value of R, we can now choose z so small that the difference

$$|\Omega(z,R) - \Omega(0,R)|$$

becomes less than ε. From this it follows that

$$| \Omega(z,0) - L| < C\varepsilon \quad \text{für} \quad |z| < z_0;$$

hence,

$$\lim_{z \to 0} \Omega(z, 0) = L,$$

i.e.,

$$\frac{d}{dn} \int_{(\Sigma)} (\mu - \mu_0) \frac{\cos(rN)}{r^2} d\sigma$$

tends to a well-defined limit as the point M on the normal N_0 to (S) at M_0 approaches M_0. This is what was required to prove.

§III. A Theorem on the Second Derivatives of the Newtonian Potential

Theorem. *If the density of a Newtonian potential is defined and H-continuous in a finite region (D) bounded by a finite number of closed* LYAPUNOV *surfaces, then the second derivatives of the potential are H-continuous in (D) and outside of (D).*

Proof. We divide the proof into two parts: (1) the proof of the H-continuity of the second derivatives in the region (D) which we shall denote by (D_i) and (2) the proof of the H-continuity of the second derivatives in (D_e).

(1) To begin with we recall that if in formula (61) of II, §17 one puts $\mu = 1$ then the formula

$$\frac{\partial}{\partial x} \int_{(D)} \frac{d\tau_2}{r_{20}} = - \int_{(S)} \cos(Nx) \frac{d\sigma_2}{r_{20}} \tag{1}$$

follows; this formula is valid whether the point M_0 lies in (D_i) or in (D_e).

The existence of the second derivatives of a Newtonian potential with H-continuous density in (D_i) was proved in II, §14; there the explicit form of these derivatives was also given. We now make use of the remark at the end of II, §14 and choose the region (D_0) to be the entire region (D). Then, for example, we obtain for $\frac{\partial^2 P}{\partial x^2}$ the expression

$$\frac{\partial^2 P}{\partial x^2} = \mu_0 \frac{\partial^2}{\partial x^2} \int_{(D)} \frac{d\tau_2}{r_{20}} + \int_{(D)} (\mu - \mu_0) \frac{\partial^2 \frac{1}{r_{20}}}{\partial x^2} d\tau_2, \tag{2}$$

which using formula (1) becomes

$$\frac{\partial^2 P}{\partial x^2} = - \mu_0 \frac{\partial}{\partial x} \int_{(S)} \cos(Nx) \frac{d\sigma_2}{r_{20}} + \int_{(D)} (\mu - \mu_0) \frac{\partial^2 \frac{1}{r_{20}}}{\partial x^2} d\tau_2. \tag{3}$$

From the theorem in II, §7 we can conclude that the right-hand side of (1), as the potential of a simple layer with H-continuous density $\cos(Nx)$, has

H-continuous first derivatives in (D_i) and in (D_e). It follows that the first summand on the right-hand side of (3) is an H-continuous function in (D_i), for it is the product of two H-continuous functions and a bounded function. It remains to prove the H-continuity of the second summand on the right-hand side of (3). For brevity we introduce the following notation:

$$\frac{\partial^2 \frac{1}{r_{20}}}{\partial x^2} = f(M_0, M)\,, \tag{4}$$

where $M(\xi,\eta,\zeta)$ is the integration point. Thus, we shall prove the H-continuity of the integral

$$\varphi(M_0) = \int_{(D)} (\mu - \mu_0) f(M_0, M)\, d\tau\ . \tag{5}$$

Now let M_0 and M_1 be two points of the region (D_i) a distance δ apart. Let $(D,2\delta)$ be the subregion of (D) contained in the interior of the sphere (2δ) of radius 2δ about the point M_0. We have the following equation:

$$\begin{aligned} -\varphi(M_0) &= \int_{(D)} (\mu - \mu_1) f(M_1, M)\, d\tau - \int_{(D)} (\mu - \mu_0) f(M_0, M)\, d\tau \\ &= \int_{(D,2\delta)} (\mu - \mu_1) f(M_1, M)\, d\tau - \int_{(D,2\delta)} (\mu - \mu_0) f(M_0, M)\, d\tau \\ &\quad + \int_{(D-(D,2\delta))} [(\mu - \mu_1) f(M_1, M) - (\mu - \mu_0) f(M_0, M)]\, d\tau . \end{aligned} \tag{6}$$

The first and second integrals on the right-hand side of equation (6) are less in absolute value than numbers of the form $cA\delta^\lambda$, for it is clear that $|f(M_0,M)| < \frac{4}{r_0^3}$ and $|f(M_1,M)| < \frac{4}{r_1^3}$ where r_0 and r_1 are the distances of the point M from the points M_0 and M_1 respectively. One can thus estimate the absolute value of the second integral as follows:

$$\begin{aligned} \left| \int_{(D,2\delta)} (\mu - \mu_0) f(M_0, M)\, d\tau \right| &< 4A \int_{(D,2\delta)} r_0^\lambda \frac{1}{r_0^3}\, d\tau \\ &\leq 4A \int_{(2\delta)} r_0^{\lambda-3}\, d\tau = \frac{16\pi \cdot 2^\lambda}{\lambda} A\delta^\lambda. \end{aligned}$$

A similar estimate is obtained for the first integral by noting that $(D,2\delta)$ is contained in the sphere of radius 3δ about M_1. It remains to estimate the last integral on the right-hand side of (6). For this we note that outside the sphere (2δ) the inequality

$$\frac{1}{2} r_0 < r' < \frac{3}{2} r_0$$

holds, where r' denotes the distance of the point M from an arbitrary point of the segment M_0M_1.

Moreover, from (4):

$$f(M_1, M) - f(M_0, M)$$
$$= \left(\frac{\partial^3 \frac{1}{r}}{\partial x^3}\right)_{M'} (x_1 - x_0) + \left(\frac{\partial^3 \frac{1}{r}}{\partial x^2 \partial y}\right)_{M'} (y_1 - y_0) + \left(\frac{\partial^3 \frac{1}{r}}{\partial x^2 \partial z}\right)_{M'} (z_1 - z_0);$$

M' is here some point of the segment $M_0 M_1$. From this it follows that

$$|f(M_1, M) - f(M_0, M)| < \delta \frac{c}{r'^4} < \frac{16 c \delta}{r_0^4}.$$

Hence,

$$|(\mu - \mu_1) f(M_1, M) - (\mu - \mu_0) f(M_0, M)|$$
$$\leq |(\mu - \mu_0)[f(M_1, M) - f(M_0, M)]| + |(\mu_0 - \mu_1) f(M_1, M)|$$
$$< \frac{A r_0^\lambda \cdot 16 c \delta}{r_0^4} + A \delta^\lambda \frac{32}{r_0^3}.$$

Let R be the diameter of the region (D). From the last inequality we can then conclude that the absolute value of the third integral on the right-hand side of (6) is no greater than

$$4\pi \left[16 c A \delta \int_{2\delta}^{R} r_0^{\lambda - 4} r_0^2 \, d r_0 + 32 A \delta^\lambda \int_{2\delta}^{R} r_0^{-3} r_0^2 \, d r_0 \right]$$
$$= A \left[\frac{64 \pi c}{1 - \lambda} \left(2^{\lambda - 1} \delta^\lambda - \frac{\delta}{R^{1-\lambda}} \right) + 128 \pi \delta^\lambda \ln \frac{R}{2\delta} \right] < c_1 A \delta^{\lambda'}.$$

The function $\varphi(M_0)$ is thus H-continuous in (D_i).

This completes the proof of the first part of the theorem, that is the part in which the H-continuity of the second derivatives of our potential in (D_i) is asserted.

Before beginning the second part of the proof, we note that in studying the difference (6) it would have been permissible to assume that the points M_0 and M_1 belonged to the boundary (S) of the region (D_i). Thus, $\varphi(M_0)$ is an H-continuous function on (S). We shall make use of this fact in the second part of the proof.

(2) We shall now prove that the second derivatives of a Newtonian potential with H-continuous density are H-continuous in (D_e). For this we note that the function

$$\frac{\partial^2}{\partial x^2} \int_{(D)} \frac{\mu}{r_0} \, d\tau$$

is harmonic outside (D) in every region which together with its boundary belongs to the interior of (D_e).

We shall have proved the theorem when we have shown the following: 1. The second derivative in question has a well-defined, finite limit $\psi(M_1)$ as the point M_0 approaches a boundary point M_1 of (D_e) in such a manner that M_0 always

remains in (D_e); 2. This limit $\psi(M_1)$ is an H-continuous function on (S). From the uniqueness of the solution of the DIRICHLET problem the second derivative of the Newtonian potential then coincides with the solution of the DIRICHLET problem for a given H-continuous function on (S). Such a solution is however, as we have seen, an H-continuous function in (D_e).

We have thus to prove that the limit $\psi(M_1)$ exists and is H-continuous on (S).

Let M_1 be a point of the surface (S). Let the point M_0 be in (D_e) on the normal N_1 at M_1 at a distance δ from M_1. We denote the value of μ at the point M_1 by μ_1. We then find using formula (1):

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\int\limits_{(D)}\frac{\mu}{r_0}\,d\tau &= \frac{\partial^2}{\partial x^2}\left(\mu_1\int\limits_{(D)}\frac{d\tau}{r_0}+\int\limits_{(D)}(\mu-\mu_1)\frac{d\tau}{r_0}\right)\\ &= -\mu_1\frac{\partial}{\partial x}\int\limits_{(S)}\frac{\cos(Nx)}{r_0}\,d\sigma+\int\limits_{(D)}(\mu-\mu_1)f(M_0,M)\,d\tau \qquad (7)\\ &= \mu_1\beta(M_0)+\alpha(M_1,\delta).\end{aligned}$$

As already mentioned, the function

$$\beta(M_0) = -\frac{\partial}{\partial x}\int\limits_{(S)}\frac{\cos(Nx)}{r_0}\,d\sigma$$

is H-continuous in (D_i) and in (D_e). Its limit values (from (D_e)) on (S) form an H-continuous function which we denote by $\beta(M_1)$. Moreover,

$$|\beta(M_0)-\beta(M_1)| < c\delta^{\lambda'}.$$

Thus, as $\delta \to 0$ the first summand in (7) has limit

$$\mu_1\beta(M_1).$$

We shall show that the second summand $\alpha(M_1,\delta)$ converges to

$$\varphi(M_1) = \int\limits_{(D)}(\mu-\mu_1)f(M_1,M)\,d\tau \qquad (8)$$

when M_0 approaches M_1 along the normal N_1. That $\varphi(M_1)$ as a function of M_1 is H-continuous was already proved in the first part.

Let $(D,2\delta)$ be the subregion of (D) lying inside the sphere of radius 2δ about M_1, where we assume that $\delta < \frac{d}{2}$. Then

$$\begin{aligned}
\alpha(M_1, \delta) - \varphi(M_1) \\
&= \int\limits_{(D)} (\mu - \mu_1) f(M_0, M)\, d\tau - \int\limits_{(D)} (\mu - \mu_1) f(M_1, M)\, d\tau \\
&= \int\limits_{(D,\,2\delta)} (\mu - \mu_1) f(M_0, M)\, d\tau - \int\limits_{(D,\,2\delta)} (\mu - \mu_1) f(M_1, M)\, d\tau \qquad (9)\\
&\quad + \int\limits_{(D-(D,\,2\delta))} (\mu - \mu_1)\, [f(M_0, M) - f(M_1, M)]\, d\tau .
\end{aligned}$$

As we have seen above, the second and third integrals on the right-hand side of (9) are less in absolute value than numbers of the form $cA\delta^\lambda$.

To estimate the first integral we note that since

$$r_1 \leq r_0 + \delta$$

the following inequality is valid:

$$|\mu - \mu_1| < A r_1^\lambda \leq A(r_0 + \delta)^\lambda < 2^\lambda A(r_0^\lambda + \delta^\lambda).$$

From this it follows that the absolute value of the first integral on the right-hand side of (9) is not greater than

$$\begin{aligned}
4 \cdot 2^\lambda A \int\limits_{(D,\,2\delta)} (r_0^\lambda + \delta^\lambda) \frac{d\tau}{r_0^3} &= 4 \cdot 2^\lambda A \left[\int\limits_{(D,\,2\delta)} r_0^{\lambda-3}\, d\tau + \delta^\lambda \int\limits_{(D,\,2\delta)} \frac{d\tau}{r_0^3} \right] \\
&< 16\pi \cdot 2^\lambda A \int_0^{3\delta} r_0^{\lambda-1}\, d r_0 + 4 \cdot 2^\lambda A\, \delta^\lambda \int\limits_{(D,\,2\delta)} \frac{d\tau}{r_0^3} \\
&= \delta^\lambda A \left[\frac{16\pi \cdot 6^\lambda}{\lambda} + 4 \cdot 2^\lambda \int\limits_{(D,\,2\delta)} \frac{d\tau}{r_0^3} \right] < c A \delta^\lambda;
\end{aligned}$$

indeed, as was just proved,

$$\int\limits_{(D,\,2\delta)} \frac{d\tau}{r_0^3}$$

is bounded for all values δ which are not greater than $\frac{d}{2}$.

Let (ξ,η,ζ) be a local coordinate system with origin at the point M_1 and ζ axis oriented toward the point M_0. It is clear that $(D,2\delta)$ lies in the region bounded by the surfaces

$$\zeta = -2\delta, \qquad \varrho = \sqrt{\xi^2 + \eta^2} = 2\delta, \qquad \zeta = b\varrho^{1+\lambda} .$$

Since $r_0 = \sqrt{\varrho^2 + (\delta - \zeta)^2}$, we now obtain the following estimate:

$$\int\limits_{(D,2\delta)} \frac{d\tau}{r_0^3} < \int\limits_0^{2\pi} \int\limits_0^{2\delta} \varrho \int\limits_{-2\delta}^{b\varrho^{1+\lambda}} \frac{d\zeta}{(\sqrt{\varrho^2+(\delta-\zeta)^2})^3} \, d\varrho \, d\varphi$$

$$= 2\pi \int\limits_0^{2\delta} \frac{1}{\varrho} \left[\frac{b\varrho^{1+\lambda}-\delta}{\sqrt{\varrho^2+(b\varrho^{1+\lambda}-\delta)^2}} + \frac{3\delta}{\sqrt{\varrho^2+9\delta^2}} \right] d\varrho$$

$$= 2\pi \int\limits_0^{2\delta} \frac{1}{\varrho} \left[\frac{3\delta}{\sqrt{\varrho^2+9\delta^2}} - 1 \right] d\varrho + 2\pi \int\limits_0^{2\delta} \frac{1}{\varrho} \left[1 - \frac{\delta - b\varrho^{1+\lambda}}{\sqrt{\varrho^2+(\delta-b\varrho^{1+\lambda})^2}} \right] d\varrho$$

$$= -2\pi \int\limits_0^2 \frac{x\,dx}{\sqrt{x^2+9}\,[3+\sqrt{x^2+9}]}$$

$$+ 2\pi \int\limits_0^2 \frac{x\,dx}{\sqrt{x^2+(1-b\delta^\lambda x^{1+\lambda})^2}\,[\sqrt{x^2+(1-b\delta^\lambda x^{1+\lambda})^2}+(1-b\delta^\lambda x^{1+\lambda})]}$$

$$< 2\pi \int\limits_0^2 \frac{dx}{\sqrt{x^2+(1-b\delta^\lambda x^{1+\lambda})^2}+(1-b\delta^\lambda x^{1+\lambda})} \, .$$

But the last integral is a bounded and continuous function of δ in an arbitrary neighborhood of the point $\delta = 0$. We thus find that

$$|\alpha(M_1, \delta) - \varphi(M_1)| < cA\delta^{\lambda'} . \tag{10}$$

If then M_0 approaches the point M_1 of (S) along the normal N_1 to (S) at M_1, then

$$\omega(M_0) = \left(\frac{\partial^2 P}{\partial x^2}\right)_{M_0} = \frac{\partial^2}{\partial x^2} \int\limits_{(D)} \frac{\mu}{r_0} \, d\tau$$

approaches the limit

$$\psi(M_1) = \mu_1 \beta(M_1) + \varphi(M_1); \tag{11}$$

where

$$|\omega(M_0) - \psi(M_1)| < cA\delta^{\lambda'} . \tag{12}$$

Since μ_1, $\beta(M_1)$, and $\varphi(M_1)$ are H-continuous on (S), it follows moreover that

$$|\psi(M_1) - \psi(M')| < c_1 A \delta_1^{\lambda'}, \tag{13}$$

where M' is a point of (S) a distance δ_1 from M_1.

From (12) and (13) it follows easily that (12) holds for any arbitrary point M_0 in (D_e) which is a distance less than δ from M_1. Indeed, the distance of the

point M_0 from the nearest point M' on (S) is no greater than δ and hence the distance δ_1 of the point M_1 from M' is no greater than 2δ. Since M_0 lies on the normal N' to (S) at the point M', it follows from this that

$$|\omega(M_0) - \psi(M_1)| \leq |\omega(M_0) - \psi(M')| + |\psi(M') - \psi(M_1)|$$
$$< cA\delta^{\lambda'} + c_1 A(2\delta)^{\lambda'} = c_2 A\delta^{\lambda'},$$

as was to be shown. Thus, the second derivative $\dfrac{\partial^2 P}{\partial x^2}$ has the well-defined limit $\psi(M_1)$, and this is an H-continuous function on (S). This completes the proof of the theorem.

§IV. The Direct Values of the Double-Layer Potential and of the Normal Derivative of the Simple-Layer Potential on a Surface L_k

We now prove Theorems 3 and 4 of II, §21.

Theorem 3. *If* $(S) \in L_{l+2}(B,\lambda)$ *and* $\mu \in H(l,A,\lambda)$ $(l \geqq 0)$ *on* (S), *then* $\overline{W}[\mu] \in H(l+1,cA,\lambda')$ *on* (S).

Theorem 4. *If* $(S) \in L_{l+2}(B,\lambda)$ *and* μ $H(l,A,\lambda)$ $(l \geqq 0)$ *on* (S), *then* $\dfrac{dV[\mu]}{dn}$ $\in H(l+1,cA,\lambda')$ *on* (S).

Proof of Theorem 3. Let M_0 be some point of the surface (S); let (ξ,η,ζ) be a local coordinate system about M_0. Let the radius d_0 of the circle (Λ_0) about M_0 in the (ξ,η) plane be so small that the circle (Λ_1) of radius $2d_0$ in the (ξ,η) plane concentric with (Λ_0) is contained in the projection of the subregion (Σ) of (S) onto the (ξ,η) plane.[1]

[1] For the notation (Σ), (Σ_0), etc. see I of this Appendix (Trans.).

We shall start with the facts that in (Λ_1) $\mu(\xi,\eta) \in H(l,A,\lambda)$ and $F(\xi,\eta) \in H(l+2,B,\lambda)$ and show that $\overline{W}[\mu] \in H(l+1,cA,\lambda')$ in (Λ_0).

We have:

$$\overline{W}[\mu] = \int\limits_{(S-\Sigma)} \mu(2)\frac{\cos(r_{21}N_2)}{r_{12}^2}\,d\sigma_2 + \int\limits_{(\Sigma)} \mu(2)\frac{\cos(r_{21}N_2)}{r_{12}^2}\,d\sigma_2. \tag{1}$$

The integral over $(S - \Sigma)$ is a function of ξ, η, and ζ which in some region containing the surface (Σ_0) has bounded and continuous derivatives of arbitrary order with respect to ξ, η, and ζ. If we replace ζ by $F(\xi,\eta)$ we obtain the value of the first integral on (Σ_0). Since $F(\xi,\eta)$ has derivatives with respect to ξ and η up to order $l+2$, this is also the case for the first integral; hence, the

first integral belongs to the class $H(l+1,c_1A,1)$ in (A_0) and hence to the class $H(l+1,cA,\lambda')$.

We denote the coordinates of the point M_1 by ξ,η,ζ and those of the integration point M_2 by x,y,z. The relations

$$r_{12}=\sqrt{(x-\xi)^2+(y-\eta)^2+[F(x,y)-F(\xi,\eta)]^2},$$

$$\cos(r_{12}N_2)=\frac{F(\xi,\eta)-F(x,y)+(x-\xi)F'_\xi(x,y)+(y-\eta)F'_\eta(x,y)}{r_{12}}\cos(N_2\zeta),$$

then obtain, for

$$\cos(r_{12}\xi)=\frac{\xi-x}{r_{12}},\qquad \cos(r_{12}\eta)=\frac{\eta-y}{r_{12}},$$

$$\cos(r_{12}\zeta)=\frac{F(\xi,\eta)-F(x,y)}{r_{12}},\qquad \cos(N_2\zeta)=\frac{1}{\sqrt{1+F_\xi'^2+F_\eta'^2}},$$

$$\cos(N_2\xi)=-F'_\xi(x,y)\cos(N_2\zeta),$$

$$\cos(N_2\eta)=-F'_\eta(x,y)\cos(N_2\zeta).$$

Hence,

$$\int\limits_{(\Sigma)}\mu(2)\frac{\cos(r_{12}N_2)}{r_{12}^2}\,d\sigma_2$$
$$=\iint\limits_{(A)}\mu(x,y)\frac{F(\xi,\eta)-F(x,y)+(x-\xi)F'_\xi(x,y)+(y-\eta)F'_\eta(x,y)}{\{\sqrt{(x-\xi)^2+(y-\eta)^2+[F(x,y)-F(\xi,\eta)]^2}\}^3}\,dxdy. \tag{2}$$

Let $\omega(r)$ be a function which has continuous derivatives up to order $l+3$ for all $r\geqq 0$ and which is equal to one for $r\leqq\frac{3}{2}d_0$ and to zero for $r\geqq 2d_0$. We put

$$\mu_1(x,y)=\mu(x,y)\,\omega\left(\sqrt{x^2+y^2}\right),$$
$$F_1(x,y)=F(x,y)\,\omega\left(\sqrt{x^2+y^2}\right),$$

where we shall assume that $\mu_1(x,y)$ and $F_1(x,y)$ are defined in the entire (x,y) plane and are equal to zero outside (A_1).

In the circle $\sqrt{x^2+y^2}\leqq\frac{3}{2}d_0$ it is clear that $\mu(x,y)=\mu_1(x,y)$ and $F(x,y)=F_1(x,y)$. Therefore, integral (2) differs from the integral obtained upon replacing $\mu(x,y)$ by $\mu_1(x,y)$ and $F(x,y)$ by $F_1(x,y)$ by an integral which extends over the subregion outside this circle. Inside the circle (A_0) this last integral also belongs to the class $H(l+1,cA,\lambda')$.

Just as $\mu(x,y)$ and $F(x,y)$, the functions $\mu_1(x,y)$ and $F_1(x,y)$ also have continuous derivatives up to order l and $l+2$ respectively in (A_1) and there belong to the class $H(l,cA,\lambda)$ and the class $H(l+2,cB,\lambda)$ respectively, since $\omega(\sqrt{x^2+y^2})$ has these properties. The functions $\mu_1(x,y)$ and $F_1(x,y)$ moreover have the properties mentioned in the entire (x,y) plane, for on the boundary of

the disk (Λ_1) and outside it μ_1 and F_1 and their derivatives up to order l and $l + 2$ respectively are equal to zero. To prove our theorem it suffices to show that

$$\varphi(\xi, \eta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu_1(x, y) \frac{F_1(\xi, \eta) - F_1(x, y) + (x - \xi) F'_{1\xi}(x, y) + (y - \eta) F'_{1\eta}(x, y)}{\{\sqrt{(x - \xi)^2 + (y - \eta)^2 + [F_1(x, y) - F_1(\xi, \eta)]^2}\}^3} \, dx \, dy$$

belongs to the class $H(l + 1, cA, \lambda')$ in (Λ_0).

In place of $\mu_1(x,y)$ and $F_1(x,y)$ we shall henceforth write $\mu(x,y)$ and $F(x,y)$ and again take the region of integration to be the entire (x,y) plane.

We go over to polar coordinates with pole the point $M_1(\xi,\eta)$ and put

$$x = \xi + \varrho \cos \Theta, \quad y = \eta + \varrho \sin \Theta .$$

We then obtain:

$$\varphi(\xi, \eta) = \int_0^{2\pi} \int_0^{\infty} \mu(\xi + \varrho \cos \Theta, \eta + \varrho \sin \Theta) K(\xi, \eta; \varrho, \Theta) \, d\varrho \, d\Theta \tag{3}$$

with

$$\left. \begin{aligned} &K(\xi, \eta; \varrho, \Theta) = \varrho \frac{F(M_1) - F(M_2) + \varrho \cos \Theta \cdot F'_\xi(M_2) + \varrho \sin \Theta \cdot F'_\eta(M_2)}{\{\sqrt{\varrho^2 + [F(M_2) - F(M_1)]^2}\}^3}, \\ &M_1(\xi, \eta), \ M_2(\xi + \varrho \cos \Theta, \eta + \varrho \sin \Theta). \end{aligned} \right\} \tag{4}$$

We shall examine the function $K(\xi,\eta; \varrho,\Theta)$ more closely. We have:

$$\begin{aligned} F(M_2) - F(M_1) &= \int_0^1 \frac{d}{dt} F(\xi + t \varrho \cos \Theta, \eta + t \varrho \sin \Theta) \, dt \\ &= \int_0^1 [F'_\xi(M) \varrho \cos \Theta + F'_\eta(M) \varrho \sin \Theta] \, dt \\ &= \varrho \int_0^1 [F'_\xi(M) \cos \Theta + F'_\eta(M) \sin \Theta] \, dt , \end{aligned}$$

where M denotes the point with coordinates $\xi + t\varrho \cos \Theta, \eta + t\varrho \sin \Theta$.

The function

$$\Psi_1(\xi, \eta; \varrho, \Theta) = \frac{F(M_2) - F(M_1)}{\varrho} = \int_0^1 [F'_\xi(M) \cos \Theta + F'_\eta(M) \sin \Theta] \, dt$$

therefore has continuous derivatives with respect to all its arguments ξ,η,ϱ,Θ up to order $l + 1$.

Integrating by parts, we obtain further:

$$\begin{aligned}
F(M_2) - F(M_1) \\
&= \int_0^1 \frac{d}{dt} F(M)\,dt = t\frac{d}{dt}F(M)\Big|_0^1 - \int_0^1 t\frac{d^2}{dt^2}F(M)\,dt \\
&= F'_\xi(M_2)\,\varrho\cos\Theta + F'_\eta(M_2)\,\varrho\sin\Theta \\
&\quad - \varrho^2\int_0^1 t\,[F''_{\xi\xi}(M)\cos^2\Theta + 2F''_{\xi\eta}(M)\cos\Theta\sin\Theta + F''_{\eta\eta}(M)\sin^2\Theta]\,dt.
\end{aligned}$$

From this it follows that the function

$$\begin{aligned}
\Psi_2(\xi,\eta;\varrho,\Theta) &= \frac{F(M_1) - F(M_2) + \varrho\cos\Theta F'_\xi(M_2) + \varrho\sin\Theta F'_\eta(M_2)}{\varrho^2} \\
&= \int_0^1 t\,[F''_{\xi\xi}(M)\cos^2\Theta + 2F''_{\xi\eta}(M)\cos\Theta\sin\Theta + F''_{\eta\eta}(M)\sin^2\Theta]\,dt
\end{aligned}$$

has continuous derivatives with respect to all the arguments ξ,η,ϱ,Θ up to order l.

Since

$$K(\xi,\eta;\varrho,\Theta) = \varrho\frac{\varrho^2\Psi_2(\xi,\eta;\varrho,\Theta)}{\varrho^3[\sqrt{1+\Psi_1^2(\xi,\eta;\varrho,\Theta)}]^3} = \frac{\Psi_2(\xi,\eta;\varrho,\Theta)}{[\sqrt{1+\Psi_1'^2(\xi,\eta;\varrho,\Theta)}]^3}$$

the function $K(\xi,\eta;\varrho,\Theta)$ has continuous derivatives up to order l with respect to ξ,η,ϱ,Θ. The function

$$\mu(\xi + \varrho\cos\Theta, \eta + \varrho\sin\Theta)$$

has this same property.

The derivative of order l of $\varphi(\xi,\eta)$ with respect to ξ and η is therefore a certain linear combination of a finite number of integrals of the type

$$\left.\begin{aligned}
&\int_0^{2\pi}\int_0^{\infty} \mu^{(m)}(\xi + \varrho\cos\Theta, \eta + \varrho\sin\Theta)\,K^{(l-m)}(\xi,\eta;\varrho,\Theta)\,d\varrho\,d\Theta \\
&\qquad (m = 0, 1, \ldots, l),
\end{aligned}\right\} \tag{5}$$

where $\mu^{(m)}$ denotes some derivative of order m of $\mu(\xi,\eta)$ and $K^{(l-m)}$ some derivative of order $l - m$ with respect to ξ,η of $K(\xi,\eta;\varrho,\Theta)$. From this it follows that all the derivatives of the function $\varphi(\xi,\eta)$ up to order l are continuous and are bounded in absolute value by numbers of the form cA. To prove the theorem it thus suffices to show that an integral of type (5) is bounded in (Λ_0) and has an H-continuous derivative of first order.

We shall now investigate the derivatives $K^{(l-m)}(\xi,\eta;\varrho,\Theta)$ in the light of formula (4) more closely.

Let $\Phi(\xi,\eta)$ be some function. The expression

$$\Phi(\xi,\eta) - \Phi(x,y) + (x-\xi)\Phi'_\xi(x,y) + (y-\eta)\Phi'_\eta(x,y)$$

we denote by $\{\Phi\}$. If we replace x and y by $\xi + \varrho\cos\Theta$ and $\eta + \varrho\sin\Theta$ respectively we obtain a function of ξ,η,ϱ,Θ which we denote by $\{\tilde{\Phi}\}$. The numerator of the fraction in (4) is thus equal to $\{\tilde{F}\}$.

It is clear that one obtains a derivative of $\{\tilde{F}\}$ with respect to ξ and η on replacing F by the corresponding derivative, i.e.,

$$\frac{\partial^q \widetilde{\{F\}}}{\partial\xi^{q_1}\partial\eta^{q_2}} = \left\{\frac{\widetilde{\partial^q F}}{\partial\xi^{q_1}\partial\eta^{q_2}}\right\} = \left\{\widetilde{F^{(q)}}\right\}.$$

The difference $\Phi(x,y) - \Phi(\xi,\eta)$ will be denoted by $[\Phi]$; the function obtained when the variables x and y in $[\Phi]$ are replaced by $\xi + \varrho\cos\Theta$ and $\eta + \varrho\sin\Theta$ will be denoted by $[\tilde{\Phi}]$. Clearly,

$$\frac{\partial^\nu \widetilde{[F]}}{\partial\xi^{\nu_1}\partial\eta^{\nu_2}} = \left[\frac{\widetilde{\partial^\nu F}}{\partial\xi^{\nu_1}\partial\eta^{\nu_2}}\right] = \left[\widetilde{F^{(\nu)}}\right].$$

We shall prove that a derivative of order k $(k \geqq 1)$ of the function $(\varrho^2 + [\tilde{F}]^2)^{-3/2}$ with respect to ξ and η is a certain linear combination of a finite number of expressions of the form

$$\left(\varrho^2 + \widetilde{[F]}^2\right)^{-\left(\frac{3}{2}+p\right)} \prod_{i=1}^{2p} \left[\widetilde{F^{(\nu_i)}}\right] \qquad \begin{pmatrix} 1 \leq p \leq k \\ 0 \leq \nu_i \leq k - p + 1 \end{pmatrix}. \tag{6}$$

Indeed, a first derivative is equal to

$$-3\left(\varrho^2 + \widetilde{[F]}^2\right)^{-\left(\frac{3}{2}+1\right)} \widetilde{[F]} \left[\widetilde{F^{(1)}}\right],$$

wherewith our assertion is proved for $k = 1$. Let $k_1 > 1$. Assuming that the assertion is true for $k = k_1 - 1$, we wish to show it is true for k_1. Differentiating (6), one obtains the following: In differentiating the first factor the number p increases by one, and the additional factor $[\tilde{F}][F^{(1)}]$ is obtained: in differentiating $\prod_{i=1}^{2p} [F^{(\nu_i)}]$ one obtains a sum of $2p$ summands of the same type where in each summand one of the ν_i is greater by one than it was initially. Therefore, every derivative of order $k_1 = k + 1$ is some linear combination of expressions either of the type

$$\left(\varrho^2 + \widetilde{[F]}^2\right)^{-\left(\frac{3}{2}+p+1\right)} \prod_{i=1}^{2(p+1)} \left[\widetilde{F^{(\nu_i)}}\right] \qquad \begin{pmatrix} 2 \leq p + 1 \leq k_1 \\ 0 \leq \nu_i \leq k - p + 1 \end{pmatrix}$$

or of the type

$$\left(\varrho^2 + \widetilde{[F]}^2\right)^{-\left(\frac{3}{2}+p\right)} \prod_{i=1}^{2p} \left[\widetilde{F^{(\nu_i)}}\right] \qquad \begin{pmatrix} 1 \leq p \leq k < k_1 \\ 0 \leq \nu_i \leq k_1 - p + 1 \end{pmatrix},$$

wherewith our assertion is proved for k_1 if it is true for $k = k_1 - 1$. Since the

assertion is true for $k = 1$, it is thus valid for all k for which the derivatives $[F^{(\nu_i)}]$ can be formed, i.e., in our case for $k \leqq l + 2$.

From what has been said it follows that each of the derivatives $K^{(l-m)}(\xi,\eta;\varrho,\Theta)$ is some linear combination of a finite number of expressions of the form

$$\varrho \cdot \{\widetilde{F^{(q)}}\} \frac{\prod_{i=1}^{2p} [\widetilde{F^{(\nu_i)}}]}{(\varrho^2 + [\widetilde{F}]^2)^{\frac{3}{2}+p}} \quad \begin{pmatrix} 0 \leq q \leq l - m \leq l \\ 0 \leq p \leq l - m \leq l \\ 0 \leq \nu_i \leq l - m \leq l \end{pmatrix}, \tag{7}$$

if the product $\prod_{i=1}^{2p} [F^{(\nu_i)}]$ for $p = 0$ is taken to be one.

We denote expression (7) by $\tilde{R}(\xi,\eta;\varrho,\Theta)$ and the function obtained on replacing in (7) $\{F^{(q)}\},[F^{(\nu_i)}],[\tilde{F}],\varrho$ by $\{F^{(q)}\},[F^{(\nu_i)}],[F], \sqrt{(x-\xi)^2 + (y-\eta)^2}$ by $R(\xi,\eta;x,y)$.

It must be shown that R is bounded. Indeed, since $q \leqq l$, $\Phi = F^{(q)} \in H(2,B,\lambda)$ and hence

$$\begin{aligned} |\{F^{(q)}\}| &= |\{\Phi\}| \\ &= |\Phi(\xi,\eta) - \Phi(x,y) + (x-\xi)\Phi'_\xi(x,y) + (y-\eta)\Phi'_\eta(x,y)| \\ &= |(x-\xi)(\Phi'_\xi(x,y) - \Phi'_\xi(\xi',\eta')) + (y-\eta)(\Phi'_\eta(x,y) - \Phi'_\eta(\xi',\eta'))| \\ &\leq 2B\varrho\varrho' \leq 2B\varrho^2, \end{aligned}$$

where (ξ',η') is some point in the interior of the segment joining the points (x,y) and (ξ,η), and ϱ' is its distance from (x,y).

Since $\nu_i \leqq l$, we have similarly $F^{(\nu_i)} \in H(2,B,\lambda)$ and hence

$$|[F^{(\nu_i)}]| = |F^{(\nu_i)}(x,y) - F^{(\nu_i)}(\xi,\eta)| \leq 2B\varrho.$$

Thus, since $\varrho^2 + [F]^2 \geqq \varrho^2$, we obtain:

$$|R| \leq \frac{\varrho \cdot 2B\varrho^2 (2B\varrho)^{2p}}{\varrho^{2p+3}} = (2B)^{2p+1} < (2B+1)^{2l+1} = c_1.$$

It follows that each of the integrals (5) is some linear combination of a finite number of expressions of the form

$$\int_0^{2\pi}\int_0^{\infty} \mu^{(m)}(\xi + \varrho\cos\Theta, \eta + \varrho\sin\Theta)\tilde{R}\,d\varrho\,d\Theta = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mu^{(m)}(x,y)\frac{R}{\varrho}\,dx\,dy \tag{8}$$

Now $m \leqq l$ and $\mu \in H(l,A,\lambda)$; therefore $\mu^{(m)} \in H(l-m,A,\lambda)$ and hence $\mu^{(m)} \in H(0,A,\lambda)$ for all $m \leqq l$. Moreover, we know that $F^{(q)}$ and $F^{(\nu_i)} \in H(2,B,\lambda)$ and find also that $F \in H(2,B,\lambda)$.

The theorem will have been proved when we have shown that an integral (8) belongs to the class $H(1,cA,\lambda')$ in (Λ_0) if $\mu^{(m)}$, $F^{(q)}$, $F^{(\nu_i)}$, and F belong to the classes mentioned and vanish on the boundary and outside the disk (Λ_1).

We first study the function R and prove the following inequalities:

$$\left|\frac{\partial}{\partial \xi}\frac{R}{\varrho}\right| < \frac{c_2}{\varrho^2}, \qquad \left|\frac{\partial}{\partial \eta}\frac{R}{\varrho}\right| < \frac{c_2}{\varrho^2}; \tag{9}$$

$$\left|\frac{\partial^2}{\partial \xi^2}\frac{R}{\varrho}\right| < \frac{c_3}{\varrho^3}, \qquad \left|\frac{\partial^2}{\partial \xi \partial \eta}\frac{R}{\varrho}\right| < \frac{c_3}{\varrho^3}, \qquad \left|\frac{\partial^2}{\partial \eta^2}\frac{R}{\varrho}\right| < \frac{c_3}{\varrho^3}. \tag{10}$$

For brevity of notation we denote $F^{(q)}$ by Φ, $F^{(\nu_i)}$ by F_i, $\prod_{i=1}^{n}[F^{(\nu_i)}]$ by ψ_n, $\sqrt{\varrho^2 + [F]^2}$ by r, and $\psi_{2p}r^{-(2p+3)}$ by Ω_p. Then from (7)

$$\frac{R}{\varrho} = \{\Phi\}\Omega_p \tag{11}$$

and hence

$$\frac{\partial}{\partial \xi}\frac{R}{\varrho} = \Omega_p\frac{\partial\{\Phi\}}{\partial \xi} + \{\Phi\}\frac{\partial \Omega_p}{\partial \xi}, \tag{12}$$

$$\frac{\partial^2}{\partial \xi^2}\frac{R}{\varrho} = \Omega_p\frac{\partial^2\{\Phi\}}{\partial \xi^2} + 2\frac{\partial \Omega_p}{\partial \xi}\frac{\partial\{\Phi\}}{\partial \xi} + \{\Phi\}\frac{\partial^2 \Omega_p}{\partial \xi^2}. \tag{12'}$$

We have already seen that

$$|\Omega_p| \le \frac{(2B)^{2p}}{\varrho^3}, \qquad |\{\Phi\}| \le 2B\varrho^2 \tag{13}$$

Further

$$\left|\frac{\partial}{\partial \xi}\{\Phi\}\right| = |\Phi'_\xi(\xi,\eta) - \Phi'_\xi(x,y)| \le 2B\varrho, \tag{14}$$

$$\left|\frac{\partial^2}{\partial \xi^2}\{\Phi\}\right| = |\Phi''_{\xi\xi}| < B. \tag{14'}$$

Differentiating Ω_p we obtain:

$$\frac{\partial \Omega_p}{\partial \xi} = -(2p+3)\frac{(\xi - x) - [F]F'_\xi(\xi,\eta)}{r^{2p+5}}\psi_{2p} + \frac{1}{r^{2p+3}}\sum_{k=1}^{2p} F'_{k\xi}\psi_{2p-1,k}$$

with

$$\psi_{2p-1,k} = \frac{\psi_{2p}}{F_k}.$$

From this follows

$$\left|\frac{\partial \Omega_p}{\partial \xi}\right| < \frac{B_1}{\varrho^4}. \tag{15}$$

In the same manner we obtain by another differentiation the estimate:

$$\left|\frac{\partial^2 \Omega_p}{\partial \xi^2}\right| < \frac{B_2}{\varrho^5}. \tag{15'}$$

Here B_1 and B_2 depend only on B and l. Inequalities (9) and (10) follow easily from (12) and (12′) and from the last inequalities (13) to (15).

If $F^{(q)} \in H(1,B,\lambda)$ and $F^{(\nu_i)} \in H(1,B,\lambda)$ we shall denote a function of type (7) by $\tilde{R}_1(\xi,\eta;\varrho,\Theta)$ or $R_1(\xi,\eta;x,y)$.

We now consider the function $\tilde{R}(\xi,\eta;\varrho,\Theta)$ defined by expression (7). Its

derivative with respect to ξ is clearly a certain linear combination of a finite number of expressions of the same type (7); in one of the expressions $F^{(q)}$ is replaced by $F^{(q+1)}$ and in another p is greater by one, while in each of the remaining expressions one of the ν_i is greater by one. From this it follows that $\frac{\partial \tilde{R}}{\partial \xi}$ is a linear combination of a finite number of expressions of type $\tilde{R}_1$.

We wish to prove the inequalities

$$|R_1| < c_4 \varrho^{\lambda-1}, \tag{16}$$

$$\left|\frac{\partial}{\partial \xi}\frac{R_1}{\varrho}\right| < c_5 \varrho^{\lambda-3}, \qquad \left|\frac{\partial}{\partial \eta}\frac{R_1}{\varrho}\right| < c_5 \varrho^{\lambda-3} \tag{16'}$$

From the argument just presented we then obtain from (16)

$$\left|\frac{\partial \tilde{R}}{\partial \xi}\right| < c_6 \varrho^{\lambda-1}. \tag{16''}$$

Indeed, since $\Phi \in H(1,B,\lambda)$, we obtain in place of (14) the estimate

$$\left|\frac{\partial}{\partial \xi}\{\Phi\}\right| = |\Phi'_\xi(\xi, \eta) - \Phi'_\xi(x, y)| < B\varrho^\lambda \tag{14''}$$

and in place of the second of inequalities (13) the inequality

$$\begin{aligned} |\{\Phi\}| &= |\Phi(\xi, \eta) - \Phi(x, y) + (x-\xi)\Phi'_\xi(x, y) + (y-\eta)\Phi'_\eta(x, y)| \\ &= |(x-\xi)(\Phi'_\xi(x, y) - \Phi'_\xi(\xi', \eta')) + (y-\eta)(\Phi'_\eta(x, y) - \Phi'_\eta(\xi', \eta'))| \\ &< 2B\varrho^{1+\lambda}. \end{aligned} \tag{13'}$$

From (12), (13), (13'), (14''), and (15) we then obtain inequality (16'). Inequality (16) follows from (13'), the first of inequalities (13), and the formula $R_1 = \varrho\{\Phi^1\}\Omega_p$.

We now consider the integral

$$\mathring{\varphi}(\xi, \eta) = \int_0^{2\pi}\int_0^{\infty} \tilde{R}(\xi, \eta; \varrho, \Theta)\, d\varrho\, d\Theta = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{\varrho} R(\xi, \eta; x, y)\, dx\, dy. \tag{17}$$

We shall show that $\mathring{\varphi}(\xi,\eta) \in H(1,c,\lambda')$, where c depends only on B and the choice of λ'. Indeed,

$$\mathring{\varphi}(\xi, \eta) = \lim_{\delta \to 0} \mathring{\varphi}_\delta(\xi, \eta)$$

with

$$\mathring{\varphi}_\delta(\xi, \eta) = \int_0^{2\pi}\int_\delta^{\infty} \tilde{R}\, d\varrho\, d\Theta$$

and

$$\frac{\partial \mathring{\varphi}_\delta}{\partial \xi} = \int_0^{2\pi}\int_\delta^{\infty} \frac{\partial \tilde{R}}{\partial \xi}\, d\varrho\, d\Theta.$$

If $0 < \delta_1 < \delta_2$ then from (16'')

$$\left|\frac{\partial \mathring{\varphi}_{\delta_1}}{\partial \xi} - \frac{\partial \mathring{\varphi}_{\delta_2}}{\partial \xi}\right| = \left|\int_0^{2\pi}\int_{\delta_1}^{\delta_2} \frac{\partial \tilde{R}}{\partial \xi}\, d\varrho\, d\Theta\right| < 2\pi c_6 \int_{\delta_1}^{\delta_2} \varrho^{\lambda-1} d\varrho$$
$$= \frac{2\pi c_6}{\lambda}(\delta_2^\lambda - \delta_1^\lambda) < \frac{2\pi c_6}{\lambda}\delta_2^\lambda;$$

from this it follows that as $\delta \to 0$ $\frac{\partial \mathring{\varphi}_\delta}{\partial \xi}$ converges uniformly to the limit

$$\int_0^{2\pi}\int_0^{\infty} \frac{\partial \tilde{R}}{\partial \xi}\, d\varrho\, d\Theta \tag{18}$$

which is the derivative of integral (17) with respect to ξ. Hence (18) is equal to $\frac{\partial \mathring{\varphi}}{\partial \xi}$.

It must now be shown that the integral (18) belongs to the class $H(0,c,\lambda')$. Since $\frac{\partial \tilde{R}}{\partial \xi}$ is a linear combination of a finite number of functions $\tilde{R}_1$, it suffices to show that the integral

$$\int_0^{2\pi}\int_0^{\infty} \tilde{R}_1 d\varrho\, d\Theta = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{R_1}{\varrho}\, dx\, dy \tag{19}$$

belongs to the class $H(0,c,\lambda')$.

Let $M_0(\xi,\eta)$ and $M_1(\xi_1,\eta_1)$ be two points a distance δ apart. We denote the distance of the point M_1 from the integration point $M_2(x,y)$ by ϱ_1. Then

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{R_1(1,2)}{\varrho_1}\, dx\, dy - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{R_1(0,2)}{\varrho}\, dx\, dy &= \iint_{\varrho \le 2\delta} \frac{R_1(1,2)}{\varrho_1}\, dx\, dy \\ - \iint_{\varrho \le 2\delta} \frac{R_1(0,2)}{\varrho}\, dx\, dy &+ \iint_{\varrho > 2\delta}\left[\frac{R_1(1,2)}{\varrho_1} - \frac{R_1(0,2)}{\varrho}\right] dx\, dy.\end{aligned} \tag{20}$$

Because of inequality (16),

$$\iint_{\varrho \le 2\delta} \frac{R_1(0,2)}{\varrho}\, dx\, dy \le 2\pi c_4 \int_0^{2\delta} \varrho^{\lambda-1} d\varrho = 2\pi c_4 \frac{(2\delta)^\lambda}{\lambda}.$$

Since the circle $\varrho \leqq 2\delta$ is contained in the circle $\varrho_1 \leqq 3\delta$, we obtain for the first integral on the right-hand side of (20) similarly the estimate $2\pi c_4 \frac{(3\delta)^\lambda}{\lambda}$. Thus the sum of the first two integrals on the right-hand side of (20) is no greater than a number of the form $c\delta^\lambda$. It remains to estimate the last integral.

From inequality (9) of II, §2 the inequality

$$\frac{1}{2}\varrho < \varrho' < \frac{3}{2}\varrho$$

is valid for the region $\varrho > 2\delta$, where ϱ' is the distance of the point M_2 from an arbitrary point of the segment M_0M_1.

From the inequalities (16′)

$$\left|\frac{R_1(1,2)}{\varrho_1} - \frac{R_1(0,2)}{\varrho}\right| = \left|\left(\frac{\partial}{\partial \xi}\frac{R_1}{\varrho}\right)_{M'}(\xi_1 - \xi) + \left(\frac{\partial}{\partial \eta}\frac{R_1}{\varrho}\right)_{M'}(\eta_1 - \eta)\right|$$
$$\leq \delta\frac{2c_6}{\varrho'^{3-\lambda}} < \delta\frac{2c_6 2^{3-\lambda}}{\varrho^{3-\lambda}} = \frac{c\delta}{\varrho^{3-\lambda}};$$

M' here denotes some point of the segment M_0M_1.

Recalling that R_1 vanishes for all sufficiently large ϱ, e.g., for $\varrho \geqq a$, we find:

$$\left|\iint\limits_{\varrho > 2\delta}\left[\frac{R_1(1,2)}{\varrho_1} - \frac{R_1(0,2)}{\varrho}\right]dx\,dy\right| \leq 2\pi c\delta\int\limits_{2\delta}^{a}\frac{\varrho\,d\varrho}{\varrho^{3-\lambda}}$$
$$= 2\pi c\delta\left[\frac{(2\delta)^{\lambda-1}}{1-\lambda} - \frac{a^{\lambda-1}}{1-\lambda}\right] < c'\delta^{\lambda};$$

here it is assumed that $\lambda < 1$. If $\lambda = 1$, then we obtain for the absolute value of the last integral in (20) an upper bound of the form $2\pi c\delta \ln \frac{a}{2\delta} < c''\delta^{\lambda'}$ with $\lambda' < 1$. Hence, the left-hand side of (20) is less than a number of the form $c\delta^{\lambda'}$. The integral (19) therefore belongs to the class $H(0,c,\lambda')$, and we have $\mathring{\varphi}(\xi,\eta) \in H(1,c,\lambda')$.

We now come to the proof of the assertion that the integrals (8) belong to the class $H(1,cA,\lambda')$ in (Λ_0).

In place of $\mu^{(m)}$ we shall write simply μ, where it is assumed that $\mu \in H(0,A,\lambda)$. Let μ_0 be the value of μ at the point M_0. Further let h be some nonzero number and M_1 be the point with coordinates $(\xi + h,\eta)$. We denote the distance of the point M_1 from $M_2(x,y)$ by ϱ_1. We consider the expression

$$\frac{1}{h}\left[\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mu\frac{R(1,2)}{\varrho_1}dx\,dy - \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\mu\frac{R(0,2)}{\varrho}dx\,dy\right]$$
$$= \mu_0\cdot\frac{1}{h}\left[\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{R(1,2)}{\varrho_1}dx\,dy - \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\frac{R(0,2)}{\varrho}dx\,dy\right] \tag{21}$$
$$+\frac{1}{h}\left[\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}(\mu - \mu_0)\frac{R(1,2)}{\varrho_1}dx\,dy - \int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}(\mu - \mu_0)\frac{R(0,2)}{\varrho}dx\,dy\right].$$

If (21) has a limit as $h \to 0$, then this limit is the derivative of one of the integrals (8) with respect to ξ. Our theorem will have been proved as soon as

we have demonstrated the existence of this limit and shown that it belongs to the class $H(0,cA,\lambda')$.

The first square bracket on the right-hand side of (21) is equal to

$$\mathring{\varphi}(\xi + h, \eta) - \mathring{\varphi}(\xi, \eta);$$

hence, the ratio of this difference to h has the limit $\frac{\partial \mathring{\varphi}}{\partial \xi}$. Since $\frac{\partial \mathring{\varphi}}{\partial \xi} \in H(0,c,\lambda')$ and $\mu_0 = \mu(\xi_0,\eta_0) \in H(0,A,\lambda)$, the limit of the first summand on the right-hand side of (21) exists and belongs to the class $H(0,cA,\lambda')$. We must now prove the same for the second summand on the right-hand side of (21).

For this we first prove that the expression

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mu - \mu_0)\left\{\frac{1}{h}\left[\frac{R(1,2)}{\varrho_1} - \frac{R(0,2)}{\varrho}\right] - \frac{\partial}{\partial \xi}\frac{R(0,2)}{\varrho}\right\} dx\,dy \\
= \frac{1}{h}\iint\limits_{\varrho \leq 2|h|}(\mu - \mu_0)\frac{R(1,2)}{\varrho_1}\,dx\,dy \\
- \frac{1}{h}\iint\limits_{\varrho \leq 2|h|}(\mu - \mu_0)\frac{R(0,2)}{\varrho}\,dx\,dy \\
- \iint\limits_{\varrho \leq 2|h|}(\mu - \mu_0)\frac{\partial}{\partial \xi}\frac{R(0,2)}{\varrho}\,dx\,dy \\
+ \iint\limits_{\varrho > 2|h|}(\mu - \mu_0)\left\{\frac{1}{h}\left[\frac{R(1,2)}{\varrho_1} - \frac{R(0,2)}{\varrho}\right] - \frac{\partial}{\partial \xi}\frac{R(0,2)}{\varrho}\right\} dx\,dy
\end{aligned} \tag{22}$$

has limit zero as $h \to 0$. The terms with integrals over the circle $\varrho \leqq 2\,|\,h\,|$ can be estimated in the following manner. From inequality (9)

$$\left.\begin{aligned}
\left|\iint\limits_{\varrho \leq 2|h|}(\mu - \mu_0)\frac{\partial}{\partial \xi}\frac{R(0,2)}{\varrho}\,dx\,dy\right| \\
< 2\pi A c_2\int_0^{2|h|}\varrho^{\lambda}\frac{\varrho}{\varrho^2}\,d\varrho = \frac{2\pi c_2\cdot 2^{\lambda}}{\lambda}A|h|^{\lambda}, \\
\left|\frac{1}{h}\iint\limits_{\varrho \leq 2|h|}(\mu - \mu_0)\frac{R(0,2)}{\varrho}\,dx\,dy\right| \\
< 2\pi A c_1\cdot\frac{1}{|h|}\int_0^{2|h|}\varrho^{\lambda}\frac{1}{\varrho}\cdot\varrho\,d\varrho = \frac{4\pi c_1\cdot 2^{\lambda}}{1+\lambda}A|h|^{\lambda}.
\end{aligned}\right\} \tag{23}$$

Taking into account that the circle $\varrho \leqq 2 \mid h \mid$ is contained in the circle $\varrho_1 \leqq 3 \mid h \mid$ and the inequality

$$|\mu - \mu_0| \leq |\mu - \mu_1| + |\mu_1 - \mu_0| < A\varrho_1^\lambda + A\,|h|^\lambda$$

holds, we obtain:

$$\left|\frac{1}{h}\iint\limits_{\varrho \leq 2|h|}(\mu - \mu_0)\frac{R(1,2)}{\varrho_1}\,dx\,dy\right|$$

$$< \frac{2\pi A c_1}{|h|}\left[\int\limits_0^{3|h|}\varrho_1^\lambda\frac{1}{\varrho_1}\varrho_1 d\varrho_1 + |h|^\lambda\int\limits_0^{3|h|}\frac{1}{\varrho_1}\varrho_1 d\varrho_1\right] = 2\pi c_1\left(\frac{3^{\lambda+1}}{\lambda+1} + 3\right)A|h|^\lambda.$$

Hence, as $h \to 0$ the first three summands on the right-hand side of (22) tend to zero. It remains to show that this is also the case for the integral over the region $\varrho > 2 \mid h \mid$.

The expression

$$\frac{1}{h}\left[\frac{R(1,2)}{\varrho_1} - \frac{R(0,2)}{\varrho}\right]$$

is equal to the value of the derivative of $\frac{R}{\varrho}$ with respect to ξ at a certain point $M'(\xi + \Theta h,\eta)$ with $0 < \Theta < 1$; hence, the expression in braces in the fourth integral on the right-hand side of (22) is the difference of the values of $\frac{\partial}{\partial\xi}\left(\frac{R}{\varrho}\right)$ at the points M' and M_0 and according to the first of inequalities (10) can be estimated by the quantity

$$\frac{c_3|h|}{\varrho''^3},$$

where ϱ'' is the distance of the point M_2 from some point M'' of the segment M_0M'.

Since from inequality (9) of II, §2 the inequality $\varrho'' > \frac{\varrho}{2}$ holds for the region $\varrho > 2 \mid h \mid$,

$$\left|\frac{1}{h}\left[\frac{R(1,2)}{\varrho_1} - \frac{R(0,2)}{\varrho}\right] - \frac{\partial}{\partial\xi}\frac{R(0,2)}{\varrho}\right| < \frac{8c_3|h|}{\varrho^3}. \tag{24}$$

From this one sees easily that the part of the integral (22) which extends over the region $\varrho > 2 \mid h \mid$ is less in absolute value than

$$2\pi A\cdot 8c_3|h|\cdot\int\limits_{2|h|}^{a}\varrho^\lambda\frac{1}{\varrho^3}\varrho\,d\varrho = \frac{16\pi c_3}{1-\lambda}A|h|[2^{\lambda-1}|h|^{\lambda-1} - a^{\lambda-1}] < cA|h|^\lambda.^{1)}$$

Thus, as $h \to 0$ the integral (22) has limit zero.

From what has been proved it follows that

$$\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}(\mu - \mu_0)\frac{\partial}{\partial\xi}\frac{R}{\varrho}\,dx\,dy \tag{25}$$

is the limit of the second summand on the right-hand side of (21).

It remains to show that the integral (25) belongs to the class $H(0, cA, \lambda')$.

For the proof let $M_1(\xi_1, \eta_1)$ be some point a distance δ from $M_0(\xi, \eta)$; let the value $\mu(\xi_1, \eta_1)$ be denoted by μ_1 and the distance of the point M_1 from $M_2(x, y)$ be denoted by ϱ_1.

We consider the difference

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mu-\mu_1)\left(\frac{\partial}{\partial\xi}\frac{R}{\varrho}\right)_{M_1}dx\,dy-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\mu-\mu_0)\left(\frac{\partial}{\partial\xi}\frac{R}{\varrho}\right)_{M_0}dx\,dy$$

$$=\iint_{\varrho\leq 2\delta}(\mu-\mu_1)\left(\frac{\partial}{\partial\xi}\frac{R}{\varrho}\right)_{M_1}dx\,dy-\iint_{\varrho\leq 2\delta}(\mu-\mu_0)\left(\frac{\partial}{\partial\xi}\frac{R}{\varrho}\right)_{M_0}dx\,dy \tag{26}$$

$$+\iint_{\varrho>2\delta}\left[(\mu-\mu_1)\left(\frac{\partial}{\partial\xi}\frac{R}{\varrho}\right)_{M_1}-(\mu-\mu_0)\left(\frac{\partial}{\partial\xi}\frac{R}{\varrho}\right)_{M_0}\right]dx\,dy.$$

The second and first integrals on the right-hand side of (26) are less than

$$\frac{2\pi A c_2}{\lambda}(2\delta)^{\lambda} \qquad \text{and} \qquad \frac{2\pi A c_2}{\lambda}(3\delta)^{\lambda}.$$

These estimates are obtained in the same way as the first of the estimates (23) if one notes that the disk $\varrho \leqq 2\delta$ is contained in the disk $\varrho_1 \leqq 3\delta$. To estimate the third integral on the right-hand side of (26), we note that the following inequality holds:

$$\left|(\mu-\mu_1)\left(\frac{\partial}{\partial\xi}\frac{R}{\varrho}\right)_{M_1}-(\mu-\mu_0)\left(\frac{\partial}{\partial\xi}\frac{R}{\varrho}\right)_{M_0}\right|$$

$$\leq|\mu-\mu_0|\cdot\left|\left(\frac{\partial}{\partial\xi}\frac{R}{\varrho}\right)_{M_1}-\left(\frac{\partial}{\partial\xi}\frac{R}{\varrho}\right)_{M_0}\right|+|\mu_1-\mu_0|\cdot\left|\left(\frac{\partial}{\partial\xi}\frac{R}{\varrho}\right)_{M_1}\right|$$

$$\leq A\varrho^{\lambda}\cdot\left|\left(\frac{\partial^2}{\partial\xi^2}\frac{R}{\varrho}\right)_{M'}(\xi_1-\xi_0)+\left(\frac{\partial^2}{\partial\xi\partial\eta}\frac{R}{\varrho}\right)_{M'}(\eta_1-\eta_0)\right|$$

$$+A\delta^{\lambda}\cdot\frac{c_2}{\varrho_1^2}<A\varrho^{\lambda}\cdot\frac{2\delta c_3}{\varrho'^3}+\frac{Ac_2\delta^{\lambda}}{\varrho_1^2};$$

M' is here some point of the segment M_0M_1. According to inequality (9) of II, §2, the inequalities $\varrho_1 > \frac{\varrho}{2}$ and $\varrho' > \frac{\varrho}{2}$ hold for the region $\varrho > 2\delta$; thus the difference in question is no greater in absolute value than

$$A(c_1'\delta\varrho^{\lambda-3}+c_2'\delta^{\lambda}\varrho^{-2}).$$

Thus, the absolute value of the third integral on the right-hand side of (26) does not exceed

$$A \cdot 2\pi \left(c_1' \delta \int_{2\delta}^{a} \varrho^{\lambda-3} \varrho \, d\varrho + c_2' \delta^\lambda \int_{2\delta}^{a} \varrho^{-2} \varrho \, d\varrho \right) < 2\pi A \left(c_1' \frac{2^\lambda \delta^\lambda}{1-\lambda} + c_2' \delta^\lambda \ln \frac{a}{2\delta} \right);$$

the last expression is less than a number of the form

$$c'' A \delta^{\lambda'} \qquad (0 < \lambda' < \lambda),$$

where the choice of c'' depends on the choice of λ'. This completes the proof of Theorem 3.

Proof of Theorem 4. Let M_0 be some point of the surface (S) and (ξ,η,ζ) be a local coordinate system about M_0. Let $M_1(\xi,\eta,\zeta)$ be a point of (S), and let $M_2(x,y,z)$ be the integration point.

We have:

$$\frac{dV[\mu]}{dn} = \int_{(S-\Sigma)} \mu(2) \frac{\cos(r_{21} N_1)}{r_{21}^2} d\sigma + \int_{(\Sigma)} \mu(2) \frac{\cos(r_{21} N_1)}{r_{21}^2} d\sigma.[2] \qquad (27)$$

The first integral on the right-hand side of (27) can be written in the form

$$\cos(N_1 \xi) \int_{(S-\Sigma)} \mu(2) \frac{\cos(r_{21} \xi)}{r_{21}^2} d\sigma + \cos(N_1 \eta) \int_{(S-\Sigma)} \mu(2) \frac{\cos(r_{21} \eta)}{r_{21}^2} d\sigma$$
$$+ \cos(N_1 \zeta) \int_{(S-\Sigma)} \mu(2) \frac{\cos(r_{21} \zeta)}{r_{21}^2} d\sigma$$

where the integrals are functions of ξ, η, and ζ which have bounded and continuous derivatives of arbitrary order in some region containing the surface (Σ_0). If in these functions we replace ζ by $F(\xi,\eta)$ we obtain the values of these integrals on (Σ_0) as functions of the point (ξ,η) of (Λ_0). Since $F(\xi,\eta)$ has continuous derivatives up to order $l + 2$, this also holds for the values of these integrals on (Σ_0), and hence they belong to the class $H(l + 1, cA, 1)$ in (Λ_0) and thus to the class $H(l + 1, cA, \lambda')$. Since $\cos(N_1 \xi)$, $\cos(N_1 \eta)$, and $\cos(N_1 \zeta)$ are likewise elements of this class, the first integral on the right-hand side of (27) belongs to the class $H(l + 1, cA, \lambda')$.

If we transform the second integral on the right-hand side of (27) in the same manner as the second integral on the right-hand side of (1), we obtain:

$$\int_{(\Sigma)} \mu(2) \frac{\cos(r_{21} N_1)}{r_{12}^2} d\sigma_2 = \cos(N_1 \zeta) \iint_{(\Lambda)} \mu(x, y) \sqrt{1 + F_\xi'^2(M_2) + F_\eta'^2(M_2)}$$
$$\cdot \frac{F(M_2) - F(M_1) + (\xi - x) F_\xi'(M_1) + (\eta - y) F_\eta'(M_1)}{\{\sqrt{(x-\xi)^2 + (y-\eta)^2 + [F]^2}\}^3} dx\, dy. \qquad (28)$$

Since $\cos(N_1\zeta) \in H(l+1,c,\lambda)$ in (A_0), the second integral on the right-hand side of (27) will have been shown to belong to the class $H(l+1,cA,\lambda')$ as soon as we have shown that the integral on the right-hand side of (28) belongs to this class.

Since $\mu_1(x,y) = \mu(x,y)\sqrt{1 + F'^2_\xi(M_2) + F'^2_\eta(M_2)}$ is an element of the class $H(l,cA,\lambda)$ if $\mu(x,y)$ is, it suffices to show that the integral

$$\int\int_{(A)} \mu_1(x,y) \frac{F(M_2) - F(M_1) + (\xi - x)F'_\xi(M_1) + (\eta - y)F'_\eta(M_1)}{\{\sqrt{(x-\xi)^2 + (y-\eta)^2 + [F]^2}\}^3}\, dx\, dy, \qquad (29)$$

in which $\mu_1 \in H(l,cA,\lambda)$ and $F(x,y) \in H(l+2,B,\lambda)$, belongs to the class $H(l+1,c'A,\lambda')$.

The proof here is the same as in Theorem 3. We point out one small difference.

If $\{F\}$ denotes the numerator of the fraction (29), then we obtain on differentiating with respect to ξ:

$$\frac{\partial\{F\}}{\partial\xi} = (\xi - x)F''_{\xi\xi}(M_1) + (\eta - y)F''_{\xi\eta}(M_1).$$

Thus if $F \in H(2,B,\lambda)$ the second derivative of $\{F\}$ with respect to ξ does not exist. For this reason, no inequalities of type (10) can be proved; one can, however, show that the estimate

$$\left|\left(\frac{\partial}{\partial\xi}\frac{R}{\varrho}\right)_{M_1'} - \left(\frac{\partial}{\partial\xi}\frac{R}{\varrho}\right)_{M_1}\right| < \frac{B_1\delta}{\varrho^3} + \frac{B_2\delta^\lambda}{\varrho^2}$$

holds, where $M_1'(\xi_1,\eta_1)$ and $M_1(\xi,\eta)$ are points a distance δ apart, and ϱ is the distance of the point M_1 from the point $M_2(x,y)$ lying outside the circle of radius 2δ about M_0. The last inequality makes it possible to obtain all the results which were formerly obtained with the help of (10).

Similarly, if $F \in H(1,B,\lambda)$ the derivative of $\{F\}$ with respect to ξ does not exist; one cannot form the inequality (14″) nor thus the inequalities (16′). Inequalities (16′) can however be replaced by the easily proved inequality

$$\left|\left(\frac{R_1}{\varrho}\right)_{M_1'} - \left(\frac{R_1}{\varrho}\right)_{M_1}\right| < \frac{B_1\delta^{1+\lambda} + B_2\varrho\delta^\lambda + B_3\varrho^\lambda\delta}{\varrho^3},$$

with the help of which one is able to obtain all the estimates found using the inequalities (16′).

This completes the proof of Theorem 4.

A SHORT BIOGRAPHY

The author of the present book is the respected scientist NIKOLAI MAXIMOVICH GÜNTER, member of the Academy of Sciences of the USSR and Professor at Leningrad University.

He was born December 5, 1871 (old calendar). After completing the gymnasium and his studies at the Physics-Mathematics Faculty of Petersburg University he remained at the University on the advice of A. A. MARKOV to prepare for scientific work. From that time to his death on May 4, 1941, his life, otherwise quite uneventful, was devoted entirely to science and teaching at the University and a number of other schools in Leningrad.

N. M. GÜNTER served for forty-seven years at Leningrad University, over thirty years at the Leningrad Institute for Transport Engineers, and over twenty years at the Pedagogical Institute. For a number of years he was active in teaching university courses for women (Bestushev Academy) and at the Leningrad Polytechnical Institute. Günter carried on this broad pedagogical work up to the last month of his life even though he was afflicted with a serious disease (lung cancer). It was not until two weeks before his death that his suffering forced him to give up his activities.

Even during these last two weeks Günter's entire interest and conversation were directed to questions of science and instruction. He spoke not at all of his illness. Just before his death he suddenly had to undergo an operation. He was to enter the hospital in two days, and he quickly called a session of the Scientific Council of the Mathematical Institute of Leningrad University, at which he presented his latest scientific work. This extreme devotion to duty and the interests of science were characteristic of the lifetime activity of N. M. Günter.

In special lectures at Leningrad University Günter was the first to illuminate a large number of areas of mathematics. He held the first such special lecture series in 1904 on the "Theory of Algebraic Forms." A great love of teaching was one of Günter's principal traits. His instruction was always creative. In preparing a new lecture—no matter whether a general or special lecture—he always inserted new ideas and found new approaches to the presentation of the material. Günter's lectures developed a special appreciation of mathematical exactness, rigor, and clear formulation of ideas.

A great number of our mathematicians enjoyed Günter's direct guidance

and gratefully recall the attention and extraordinary interest he showed toward the work of young scientists. While he might sharply criticize the work, he was always able to encourage the man and to convince him that the difficulties were not insurmountable and direct him toward independent creative work.

One of Günter's characteristic traits was a deep feeling of social obligation which was basic to his nature. This revealed itself in all his work as a teacher and organizer of scientific work. For many years he was Chairman of the Leningrad Mathematical Society. Under his direction the functioning of the Society, which until then had been slow and inconsequential, experienced a lively stimulus. A journal was founded and began to appear regularly.

When teaching was halted at Petersburg University in 1905, Günter organized the "Free University." One of the instructors was the late A. N. KRYLOV who presented lectures on the "Theory of Approximation" which were then later published in book form.

At Leningrad University Günter was also a tireless organizer of scientific student clubs and seminars and devoted an immense energy to these activities. To all these he contributed sound principles, organization, and enthusiasm. Thus, as head of a great scientific school and as a tireless organizer of many worthwhile undertakings, he personally made a major contribution to the flourishing of the sciences.

The great success of Günter's pedagogical activity, especially that of his special lectures, is mainly due to the fact that it was intimately connected with his own intense creative research from which he was never able to desist.

Günter used to say that the modern scientist is constantly at work from morning to night and not only when he sits behind his desk. This applies before all to himself.

At the time he studied at Petersburg University and there began his work, the tradition of P. L. CHEBYSHEV, one of the greatest mathematicians of the second half of the 19th century and founder of the famous Petersburg school of mathematics, was still very much alive. Such well known exponents of this school as A. A. MARKOV and A. N. KORKIN were Günter's teachers. In his work we thus have the brilliant continuation and development of the famous mathematical tradition of the Petersburg school. His later work in the field of mathematical physics was closely related to the work of the late Academicians A. M. LYAPUNOV and V. A. STEKLOV in whom the mathematical public of our country takes the same pride as in N. M. Günter.

Günter's early work is concerned with the general theory of ordinary and partial differential equations. This work is directly connected with two dissertations—his master's dissertation "On the Application of the Theory of Algebraic Forms to the Integration of Linear Differential Equations" which he presented in 1904 and his doctoral dissertation "On the Theory of

Characteristics of Systems of Partial Differential Equations" of the year 1915.

The general theory of linear ordinary differential equations, especially problems of applying algebraic forms to this theory and finding cases in which the integrals of the equation appear in the form of algebraic functions, was a major topic of interest in the second half of the 19th century. In Günter's master's dissertation there is an index of more than a hundred works related to the topic of the dissertation. Among these are papers of such outstanding mathematicians as FUCHS, SCHWARZ, DARBOUX, HALPHEN, etc. In the dissertation, in addition to other questions, the problem of integrating a linear equation with rational coefficients was studied for the case in which a certain form with constant coefficients constructed from unknown particular solutions is known in the form of a function of the independent variables. Sufficient conditions were given that the general integral of the equation be an algebraic function, and some cases were considered in which these conditions are not satisfied. The theory of differential equations of second order with a general algebraic integral was founded on this basis. The theory of differential equations of third order was considered from this same viewpoint. In addition to a number of specific new results, general considerations were presented in the dissertation from which results earlier obtained by FUCHS, SCHWARZ, KLEIN, HALPHEN, etc. follow immediately.

The work of Günter in the area of the general theory of partial differential equations is concerned with three different questions. The first is related to the theory of elimination and the general integrability conditions for systems of differential equations. One considers an arbitrary system of differential equations $F_i = 0$ ($i = 1, 2, \ldots, N$) with several functions $u_1, u_2, \ldots, u_k$ of several independent variables in which on the left-hand side the derivatives of u_s occur up to order n_s; if one differentiates these equations several times, then from these equations algebraic consequences may be drawn which give new relations between the derivatives of u_s of order not higher than n_s. An example of a similar type is afforded by systems of first order with one function and the POISSON bracket. The problem of the character and number of such new relations is related to the following problem from the theory of algebraic forms: Over a given basis $F_1, \ldots, F_m$ of algebraic forms of degree n_0 a module is constructed, i.e., the set of all forms of the type $\Phi = B_1F_1 + \ldots + B_mF_m$ where the B_j are forms of degree k; it is required to find all such linearly independent forms of a given degree and to form all the dependence relations of the type $\Phi \equiv 0$. This problem was solved by HILBERT in 1890. He found that all identities are linear combinations of a certain number s_0 of such identities. For sufficiently great degree $k > k_0$, he gave the number of linearly independent Φ. The theorems of HILBERT however contained only the existence proof for the numbers s_0 and k_0. The estimates

carried out by his method gave extremely large values for s_0 and k_0. Günter gave a regular procedure for the construction of all these dependencies. These questions are very difficult, and many attempts made before Günter were fruitless. He was the first to finally settle the problem in a complete fashion.

Günter's doctoral dissertation "On the Theory of Characteristics of Systems of Partial Differential Equations" belongs to the second direction of his research in the general theory of partial differential equations. In this work he presents an analytic theory of systems of partial differential equations of a very general form—the so-called RIQUIER systems, and studies the following problem usually referred to as the CAUCHY problem: It is required to find the values of the derivatives of arbitrary order of unknown functions $u_1, \ldots, u_h$ on a given surface $x_m = \psi(x_1, \ldots, x_{m-1})$ when only certain first derivatives of these functions are known. This problem is treated with exhaustive completeness. In particular, Günter gave a complete classification of the characteristics of RIQUIER systems.

Finally, it is necessary to mention his work on the analytic solutions of the equation $u_{xy} = f(x, y, u_x, \ldots, u_{yy})$. In this work Günter investigated the problem of GOURSAT on determining a solution knowing its values on two intersecting curves. The study is conducted with great completeness. By estimating the coefficients in an expansion of the unknown function obtained with the help of difference equations, Günter succeeded in giving a criterion for the existence of analytic solutions of this problem.

Günter's work on questions of mathematical physics is rich in new ideas and of broad scope and great diversity. Publication of this work began in 1922 and continued up to the last year of his life. The first great cycle of these publications consists of work which is concerned with a basic nonlinear problem of mathematical physics—the CAUCHY problem and the mixed problem for the equations of hydrodynamics. In the case of the CAUCHY problem the principal result is establishing the existence and uniqueness of the solution of the hydrodynamical equations of an ideal incompressible fluid in the presence of an external force what has a potential. It is hereby assumed that the fluid fills the entire space and that the velocity field is initially given. This field is characterized by three continuous functions which have bounded derivatives that may have jumps on passing through certain surfaces. The existence of derivatives of second order for the velocity field at the initial time point is not required. Forming the divergence of both sides of the equation of hydrodynamics and applying the POISSON formula, Günter eliminated the pressure from this equation and obtained on passing to the components of the velocity three nonlinear integral equations for these components. The right-hand sides of these equations are the time integrals of the components of the gradient of a certain Newtonian potential whose

density contains the derivatives of the velocity components in nonlinear fashion. In addition to these three equations, there are three more differential equations for the instantaneous streamlines which can be written in integral form. These six equations are written in LAGRANGE coordinates and the method of successive approximations is applied.

Most of the papers are devoted to the study of the second derivatives of the Newtonian potential with weak hypotheses on the density. The results of this research were used to study the convergence in the method of successive approximations.

The method of successive approximations for the case of the bounded problem, i.e., for the case of a fluid in a vessel which as time goes on can be displaced in a given manner and can be deformed such that the volume remains constant, is built up in quite another fashion. Here, in addition to the initial condition, there is a boundary condition on the surface of the vessel. The method of successive approximations rests in this case not on the velocity field but rather on the vorticity field which is taken as the initial condition at the initial time point. Günter first derived a formula which for a fluid in a closed vessel makes it possible to determine the velocity field from a given vorticity field. Such a formula had been given by HELMHOLTZ for unbounded space. In the case of a closed vessel the derivation of such a formula becomes considerably more complicated; it requires the solution of a NEUMANN problem and the transformation of the simple-layer potential into a double-layer potential, whereby the derivatives of this double-layer potential enter in the final formulas. The basis here of the method of successive approximations are the CAUCHY relations in which the components of the vorticity are expressed in terms of the initial values and the derivatives of the Cartesian coordinates of the fluid particles with respect to the LAGRANGE variables. As first approximation for the velocity field one takes the gradient of the harmonic function which solves the NEUMANN problem for the boundary conditions on the surface of the vessel. From this velocity field one constructs the instantaneous streamlines; the CAUCHY relations mentioned give the first approximation to the corresponding velocity field. In this manner one obtains the second approximation to the velocity field, and the procedure is repeated. The method of successive approximations leads to a velocity field which satisfies the HELMHOLTZ equations which follow from the CAUCHY relations. After this there follows an extensive investigation in which it is proved that the velocity field obtained is dynamically possible, i.e. admits the determination of a scalar pressure field from the equations of hydrodynamics. This proceeds from the proof of the fact that the velocity field expressed in terms of LAGRANGE variables has a derivative with respect to time, and this in turn has derivatives with respect to the LAGRANGE coordinates. The existence of these derivatives

makes it possible from the HELMHOLTZ equations to conclude the possibility of determining the pressure. As for the CAUCHY problem, the convergence of the method for some finite interval of time is proved.

There are certain additional complications if it is assumed that the region occupied by the fluid is multiply connected. Günter's extensive work on the mixed problem was published in the Reports of the Academy of Sciences in the form of a huge treatise consisting of six parts from 1926 to 1928.

The basic problem of hydrodynamics treated in Günter's work is mathematically related to another hydrodynamical problem which is bound historically with Petersburg University. About seventy years ago, P. L. CHEBYSHEV suggested as a topic to his young student A. M. LYAPUNOV the question of the equilibrium configuration of a rotating mass of fluid whose particles interact according to NEWTON'S law. At the same time as A. M. LYAPUNOV, POINCARÉ also occupied himself with the same problem. After encountering great difficulties in obtaining a rigorous solution, POINCARÉ reconciled himself to an approximate solution and stated that such a problem, which has an obvious physical meaning, does not require a rigorous mathematical solution. In one of his papers A. M. LYAPUNOV challenged this interpretation of POINCARÉ and said that a problem mathematically posed must also be solved with all the necessary mathematical rigor. In his basic work in the field of hydrodynamics Günter remained faithful to this legacy of A. M. LYAPUNOV.

In order to be able to carry out his work in hydrodynamics, N. M. Günter conducted a number of investigations in the field of classical potential theory. In this regard he set for himself the task, starting essentially from the work of LYAPUNOV, to develope all of potential theory in a systematic and rigorous fashion. This is the content of the monograph "La théorie du potential et ses applications aux problèmes fondamentaux de la physique mathématique" whose translation, with some completions, forms the content of the present book.

At the same time as the work on hydrodynamics, in 1925 there appeared a paper on a lemma of POINCARÉ in which Günter established that POINCARÉ had proved this lemma, which is basic to many works in the area of mathematical physics, only for regions bounded by convex surfaces. In Günter's paper this lemma is proved for the first time for the general case of a region bounded by a LYAPUNOV surface. The proof requires delicate geometric arguments related to subdividing the region into smaller subregions for which the validity of the lemma can be established.

We now pass on to the great cycle of Günter's work which is concerned with applying the method of smoothing when operating with functions having no derivatives. This work led Günter to new formulations of problems in the field of mathematical physics and to a systematic application of the

concept of a set function and the STIELTJES integral to the solution of these problems in their new formulation.

In his work on hydrodynamics Günter was repeatedly forced to work with functions which did not possess sufficient derivatives in order to be able to apply the usual methods for handling the problem. In a number of papers Günter made use of the method of smoothing in such problems. This method, which found repeated use in the work of V. A. STEKLOV, consists in replacing a function by the integral of this function over a small interval $(x, x + h)$ divided by the length of this interval. The procedure can also be applied in the case of several variables. The function obtained in this manner Günter called a STEKLOV function. Günter's first work in this direction bore the title "On operations with functions having no derivatives" (Reports of the Academy of Sciences, 1924). In this paper he was concerned primarily with the solution of the equations rot $X = A$ and grad $X = A$, where A is a given vector which as a function is continuous and X is the vector or scalar sought. It is not assumed that the components of the given vector have derivatives. In the paper a necessary and sufficient condition for the solvability of the equations is given. If one constructs a Newtonian vector potential B using the vector A as density, then for the solvability of the first of the equations it is necessary and sufficient that div B be a harmonic function and for the second equation that rot B be a harmonic vector. In this same paper Günter treated the problem in another formulation. He replaced the equations mentioned by the integral equations obtained on integrating the equations over some region and applying GAUSS' formula. The conditions mentioned above also turn out to be necessary and sufficient for the solvability of the integral equations so obtained, whereby it suffices that the components of the given vector be assumed to be merely bounded and integrable and that the components of the vector sought be continuous.

We have gone into the content of this paper in some detail for the reason that it turned out to be the starting point of Günter's fundamental examination of the proper formulation of problems of mathematical physics. The method of smoothing led him naturally to the general concept of the additive set function. In place of this concept he usually employed the idea of a mean function, which is the quotient of a set function and the measure of its domain. A central feature of the theory is the determination of the totality of those regions or sets for which such a function is defined. This also subsequently plays a role in the integral of a mean function. Günter employed the idea of the mean function for the first time in expanding a given function in terms of KORN's functions. In 1932 in the Publications of the V. A. STEKLOV Physics-Mathematics Institute Günter published a large work of about five hundred pages under the title "Sur les intégrales de STIELTJES et

leurs applications aux problèmes de la physique mathématique". The first two chapters of this work contain a systematic presentation of the theory of mean functions and the integral calculus of such functions. On this basis a theory of integral equations is developed. These investigations carried out by Günter have points in common with the work of RADON and RIESZ.

In a number of papers N. M. Günter later returned to the general theory of integral equations with STIELTJES integrals and constructed the spectral function for a wide class of such equations using the resolvent representation of MITTAG-LEFFLER. These papers have much in common with the well-known work of WEYL and CARLEMAN on the theory of singular integral equations. In one of his notes Günter shows that the theory of weighted integral equations found in the work of KNESER and LICHTENSTEIN is a special case of the theory of integral equations he developed. In a number of papers Günter carried through an exhaustive investigation of kernels of a special type—FOURIER kernels—and of integral equations with such kernels.

We have up to now spoken only about mathematical questions which are related to the theory of mean functions. For Günter problems of mathematical physics were the main stimulus to the development of this theory. In the second edition of his book "Leçons sur l'intégration . . . " LEBESGUE stressed the role that set functions play in questions of mathematical physics. The idea of point functions had heretofore been used since an extensive theory of such functions was available, while the corresponding theory of set functions had not yet been worked out.

The use of the concept of point functions led in mathematical physics to abstract ideas which did not correspond to the actual phenomena. The basic equations of mathematical physics were formulated with the help of these abstract ideas. A number of unnatural additional restrictions which one has to impose on the solutions of problems in mathematical physics have their origin in just this unsuitable formulation of the problem. It is more natural instead of speaking of the temperature at a given point to speak of the average temperature in a given region; it is more natural to speak of the average velocity over a certain time interval than to speak of the velocity at a given point of time; instead of the normal derivative at a given point of a surface on should speak of the flow through a given region of the surface etc. In several small communications Günter illuminated this point of view in an extremely clear fashion by a number of examples; these can be found in the works "La théorie des fonctions de domaines dans la physique mathématique" (Prace matematyczne, 1935), "On the smoothing of functions" (Scientific Writings of the Leningrad State University, 1937), "On certain formulations of problems in mathematical physics" (Scientific Writings of Leningrad State University, 1940). Günter did not restrict

himself to new formulations of problems but rather actually solved a number of problems of mathematical physics in the new formulation.

In the major publication of 1932 mentioned above Günter worked out a potential theory and the solution of the NEUMANN and DIRICHLET problems under the new point of view. As an example, we shall here consider the DIRICHLET problem. It is assumed that there is given on the surface S of a body D a certain mean function $\varphi(\sigma)$ of small pieces of this surface. It is required to find a function harmonic in D with the condition that its mean value on the surface element σ_1 converges to $\varphi(\sigma)$ when σ_1 goes over into the small surface element σ on S. The problem has a definite solution if $\varphi(\sigma)$ is continuous. Here only the boundary conditions are treated from the new point of view. We can bring them into the LAPLACE expression also. If we displace a region ω in the direction l we come naturally to the idea of the derivative $\varphi_e(\omega)$ of the function $\varphi(\omega)$ of the region ω in the direction l. The sum of the derivatives of second order in three perpendicular directions yields the LAPLACE expression $\Delta\varphi(\omega)$ of the set function. If $v(x)$ is a Newtonian potential whose density is some mean function and if $v(\omega)$ is the mean function of $v(x)$, then as Günter showed the generalized POISSON formula holds. In addition to this formula Günter introduced a new definition of the flow $\sigma(v)$ through the surface σ, a definition which requires no derivative of v but includes the integral of the product vK over σ where K is the mean curvature of σ. For this flow Günter set up the formula $\Delta v(\omega)\omega = \sigma(v)\sigma$. All this made it possible for him to formulate and solve the problem of heat conduction for bounded bodies on the basis of the concept of heat flow; this was presented in the last chapter of the work of 1932 mentioned above.

There is a surprising application of potential theory and of the generalized LAPLACE operator to the problem of the general form of a continuous linear functional $U\{\varphi(x)\}$ on the space of continuous functions $\varphi(x)$ of three independent variables. It was shown by RADON that the problem can be solved in the form of the STIELTJES integral of the product $\varphi(x)u(\omega)d\omega$ where $u(\omega)$ is a certain set function. In his work "Sur les opération linéaires" (Phys. Journal of the Soviet Union, 1933) Günter showed that the function $u(\omega)$ can be constructed in the following manner. One applies the functional U to the mean value of the Newtonian potential of the region ω with unit density. One then obtains a certain function of the region ω, and the LAPLACE expression of this function divided by -4π gives $u(\omega)$.

In his last papers Günter applied his general ideas to the investigation of the small vibrations of a string. This is based on the formula in which the amplitude of the string is expressed by the integral of the product of a GREEN'S function and the sum of the external and inertial forces. On replacing the acceleration by the mean change of velocity and going over

to the mean functions of the outer force and the amplitude sought, Günter arrived at an integrodifferential equation. This equation he solved under hypotheses which are fully justified by the physical meaning of the problem.

We are now experiencing a transition period in the history of mathematical physics. Through the penetration of the ideas and methods of the modern theory of functions of a real variable and functional analysis into mathematical physics our concepts of the formulation of problems, of the methods of solving these problems, and of the concept of what constitutes a solution of the problem are undergoing basic changes. This metamorphosis of mathematical physics is related to the profound new ideas which are currently arising in theoretical physics. In this new mathematical physics which is now abuilding the work of N. M. Günter will assume its due place. Characteristic of this work is the following idea which Günter formulated in his last papers. He said that by applying the method of smoothing "the problem which has as its objective the explanation of a phenomenon of the external world is partially freed of restrictions which are necessarily imposed on it by our limited tools, and nature, once freed of these restrictions, begins to reveal her secrets".

It is not only the scientific, pedagogical, and social achievements of Günter which one remembers when one thinks of him. All who were in close relationship with him will never forget this man who addressed the utmost sincerity and honor to all his work and all his relations with other men. N. M. Günter had friends, but his greatest friend was truth.

V. I. Smirnov
S. L. Sobolev

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